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# GENERALIZING THE MAHLO HIERARCHY, WITH APPLICATIONS TO THE MITCHELL MODELS

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## 1. Introduction

Most of the results appearing in this paper appeared in the author's Ph.D. thesis, which was written at the University of Colorado under the direction of William Reinhardt, whom I would like to thank for the many hours he spent discussing this material with me. In addition, I would like to thank James Baumgartner, Richard Laver, William Mitchell, Robert Solovay, and the referee for useful discussions and/or corrections involving my thesis and an earlier version of this paper. Some of this research was done while the author was a Visiting Assistant Professor at the University of Iowa.

We assume the reader is familiar with the terms closed, unbounded, and stationary for subsets of a regular cardinal. A regular cardinal $\kappa$ is called *weakly-Mahlo* if $\{\alpha<\kappa\colon \alpha \text{ is regular}\}$ is stationary in $\kappa$. The process can then be repeated to the next stage to (possibly) get a higher order Mahlo cardinal $\kappa$ such that $\{\alpha<\kappa\colon \alpha \text{ is weakly-Mahlo}\}$ is stationary, and so forth. In fact, we can give the following definition that tells us how to continue this process.

**1.1. Definition.** Let $\kappa$ be an uncountable regular cardinal. Let

$$\mathrm{NS}=\{X\subseteq\kappa\colon X \text{ is nonstationary}\}.$$

Define operations $\mathcal{M}^\alpha:\mathcal{P}(\kappa)/\mathrm{NS}\to\mathcal{P}(\kappa)/\mathrm{NS}$ by induction on $\alpha$ for $\alpha<\kappa^+$. Let $\mathcal{M}^0$ be the identity function. Let $\mathcal{M}^1([X])=[\{\alpha\in X\colon \alpha$ is uncountable and regular and $X\cap\alpha$ is stationary in $\alpha\}]$ (where $[X]$ indicates the equivalence class modulo the ideal NS). $\mathcal{M}^1$ is well-defined (see [3]) and can be iterated as follows:

If $\mathcal{M}^\beta$ has been defined, let $\mathcal{M}^{\beta+1}=\mathcal{M}^1\circ\mathcal{M}^\beta$ and if $(\mathcal{M}^\gamma\colon\gamma<\beta)$ have been defined and $a\in\mathcal{P}(\kappa)/\mathrm{NS}$, let

$$\mathcal{M}^\beta(a)=\inf\{\mathcal{M}^\gamma(a)\colon \gamma<\beta\},$$

provided $\beta < \kappa^+$ is a limit ordinal. (This infimum exists in the Boolean algebra $\mathcal{P}(\kappa)/\mathrm{NS}$ as long as $\beta < \kappa^+$.) We then let

$$m(\kappa) = \{\beta < \kappa^+ : \mathcal{M}^\beta([\kappa]) \neq \mathrm{NS}\}$$

(i.e. $m(\kappa)$ is essentially the number of times $\mathcal{M}$ must be iterated in order to get the zero of the Boolean algebra $\mathcal{P}(\kappa)/\mathrm{NS}$).

If $\kappa$ is not uncountable and regular, we extend the definition of $m$ to $\kappa$ by letting $m(\kappa) = 0$. It is easy to show that

$$m(\kappa) \geq 1 \quad \text{iff} \quad \kappa \text{ is uncountable and regular,}$$
$$m(\kappa) \geq 2 \quad \text{iff} \quad \kappa \text{ is weakly-Mahlo}$$

and so forth. If $m(\kappa) = \kappa^+$, then as in [3] we will call $\kappa$ *greatly Mahlo*. (The Mahlo operation $M$ used in [3] differed slightly from the operation $\mathcal{M}$ used here, but it is easy to check that the definition of greatly Mahlo is equivalent.) As shown in [3], if $\kappa$ is weakly compact, then $\kappa$ is greatly Mahlo.

The main purpose of the next two sections will be to generalize the definitions given here for the function $m$, which can be considered as a measure of the 'largeness' of various cardinals. However, if $\kappa$ is the first greatly Mahlo cardinal, then $m(\kappa) = \kappa^+$, whereas we also have $m(\kappa) = \kappa^+$ if $\kappa$ is supercompact. Since greatly Mahlo cardinals and supercompact cardinals have a very different 'degree of largeness', it is natural to ask whether the definition of $m$ could be altered somewhat to a function $m^*$ that would differentiate between 'small' greatly Mahlo cardinals and 'larger' ones. Some of the properties which such a function $m^*$ should have include the following:

(1) If $m(\kappa) < \kappa^+$, then $m^*(\kappa) = m(\kappa)$. (Actually, this will be violated in a minor way when we eventually define $m^*$ in Section 3, due to a change in indexing which greatly simplifies all of the definitions, but the spirit of this property will remain.)

(2) Given the existence of already well-known types of large cardinals, we should be able to find cardinals $\kappa$ such that $m^*(\kappa) > \kappa^{++}$, $m^*(\kappa) > \kappa^{+++}$ and so forth.

(3) If $\tau$ is a 'reasonable' term, such as $\tau(\kappa) = \kappa^{++} + \kappa^+$, or $\tau(\kappa) = \kappa^+ + 4$, and $m^*(\kappa) > \tau(\kappa)$, then we should have $\{\alpha < \kappa : m^*(\alpha) = \tau(\alpha)\}$ stationary in $\kappa$. (This will be true with a few well-defined exceptions, notably when $(\forall\kappa)$ ($\tau(\kappa)$ is regular).)

Before going on to these more general definitions, it is instructive to look at where the problems are. The main one is that in general infimums of subsets of $\mathcal{P}(\kappa)/\mathrm{NS}$ of size $\kappa^+$ do not necessarily exist (if they do, then by a theorem of Solovay, either $2^\kappa = 2^{\kappa^+}$ or NS is $\kappa^+$-saturated, and if NS is $\kappa^+$-saturated, then $\kappa$ is not greatly Mahlo [3], so if the GCH held, for example, these infimums could exist only where we were not interested in using them). We could try defining $\mathcal{M}^{\kappa^+}([\kappa])$ as $[\{\alpha < \kappa : \alpha \text{ is greatly Mahlo}\}]$, but this would only postpone the problem, since any attempt to repeat this procedure using only elements of

$\mathcal{P}(\kappa)/\mathrm{NS}$ is doomed to fail before $(2^\kappa)^+$ steps, since $|\mathcal{P}(\kappa)| = 2^\kappa$. In the next 2 sections we will solve this problem by using bigger sets. Finally, in Section 4 we will show how to use these Mahlo operations to define canonical inner models having many measurable cardinals.

## 2. Mahlo operations on $\mathcal{P}_\kappa\lambda$

The generalizing of the concepts closed-unbounded and stationary to subsets of $\mathcal{P}_\kappa\lambda$ is due to Jech [4]. The basic definitions are:

**2.1. Definition.** Let $\kappa > \omega$ be regular, $x$ a set of ordinals with $|x| \geq \kappa$ and $C \subseteq \mathcal{P}_\kappa x = \{y \subseteq x: |y| < \kappa\}$. $C$ is called *closed* (in $\mathcal{P}_\kappa x$) iff whenever $D \subseteq C$ is a chain under inclusion with $|D| < \kappa$, $\bigcup D \in C$. $C$ is *unbounded* (in $\mathcal{P}_\kappa x$) iff for every $y \in \mathcal{P}_\kappa x$ there is a $z \in C$ with $y \subseteq z$. $C$ is *club* iff it is both closed and unbounded. $S \subseteq \mathcal{P}_\kappa x$ is *stationary* iff $S \cap C \neq \emptyset$ for all club $C$. If $(X_\alpha: \alpha \in X)$ is a sequence of subsets of $\mathcal{P}_\kappa x$, let

$$\Delta(X_\alpha: \alpha \in x) = \{y \in \mathcal{P}_\kappa x: (\forall \alpha \in y) y \in X_\alpha\}.$$

Let $\mathrm{NS}(\kappa, x) = \{X \subseteq \mathcal{P}_\kappa x: X \text{ is not stationary}\}$. Then $\mathrm{NS}(\kappa, x)$ is an ideal on the Boolean algebra $\mathcal{P}(\mathcal{P}_\kappa x)$ and we let $\mathcal{B}(\kappa, x) = \mathcal{P}(\mathcal{P}_\kappa x)/\mathrm{NS}(\kappa, x)$ be the quotient algebra, with $[X]$ being the appropriate equivalence classes for $X \subseteq \mathcal{P}_\kappa x$. If $\{[X_\alpha]: \alpha \in x\} \subseteq \mathcal{B}(\kappa, x)$, then $\Delta\{[X_\alpha]: \alpha \in x\}$ is defined to be $[\Delta(X_\alpha: \alpha \in x)]$.

Some of the standard properties which follow immediately from the definitions are stated in the following theorem.

**2.2. Theorem.** (a) *$C \subseteq \mathcal{P}_\kappa x$ is closed iff whenever $D \subseteq C$ is directed under inclusion and $|D| < \kappa$, $\bigcup D \in C$.*

(b) *The intersection of less than $\kappa$ club subsets of $\mathcal{P}_\kappa x$ is club.*

(c) *If $C_\alpha$ is club for all $\alpha \in x$, then $\Delta(C_\alpha: \alpha \in X)$ is club.*

(d) *$\Delta\{[X_\alpha]: \alpha \in x\}$ is well-defined and is the infimum in $\mathcal{B}(\kappa, x)$ of $\{[X_\alpha]: \alpha \in x\}$.*

As noted above, the main problem is that the diagonal intersection of $\kappa^+$ subsets of $\kappa$ has no reasonable definition. However, if we use $\mathcal{P}_\kappa\lambda$ instead of $\kappa$ for some cardinal $\lambda > \kappa$, then we can use Theorem 2.2(d) to take diagonal intersections of anything less than $\lambda^+$ subsets of $\mathcal{P}_\kappa\lambda$. What we need is a corresponding version of the Mahlo operation $\mathcal{M}$ to iterate $\lambda^+$ times.

**2.3. Definition.** Let $\kappa > \omega$ be regular and let $|x| \geq \kappa$. For each $\beta < |x|^+$ define a function

$$\mathcal{M}^\beta: \mathcal{B}(\kappa, x) \to \mathcal{B}(\kappa, x)$$

by induction on $\beta$. Let $[X] \in \mathscr{B}(\kappa, x)$. Define

$$\begin{aligned}
&\mathcal{M}^0([X]) = [X],\\
&\mathcal{M}^1([X]) = [\{y \in X\colon y \cap \kappa \text{ is an uncountable regular cardinal and}\\
&\qquad\qquad X \cap \mathscr{P}_{(y\cap\kappa)}y \text{ is stationary in } \mathscr{P}_{(y\cap\kappa)}y\}],\\
&\mathcal{M}^{\beta+1}([X]) = \mathcal{M}^1(\mathcal{M}^\beta([X])),\\
&\mathcal{M}^\beta([X]) = \inf\{\mathcal{M}^\gamma([X])\colon \gamma < \beta\} \quad \text{if} \quad \beta \text{ is a limit ordinal } < |x|^+.
\end{aligned}$$

Let

$$m_x(\kappa) = \{\beta < |x|^+\colon \mathcal{M}^\beta([\mathscr{P}_\kappa x]) \neq \mathrm{NS}(\kappa, x)\}.$$

Before we can verify that this definition is well-defined, we need a lemma:

**2.4. Lemma.** *If $C \subseteq \mathscr{P}_\kappa x$ is club, then there is a club $B \subseteq C$ such that whenever $y \in B$ and $y \cap \kappa$ is an uncountable regular cardinal, then $\mathscr{P}_{(y\cap\kappa)}y \cap C$ is club in $\mathscr{P}_{(y\cap\kappa)}y$.*

**Proof.** Define a function $f\colon \mathscr{P}_\omega x \to C$ such that $y \subseteq f(y)$ and if $y \subseteq z$, $y \neq z$, then $f(y)+1 \subseteq f(z)$. This is easy to do by induction on $|y|$, using the unbounded property of $C$. Extend $f$ to $\mathscr{P}_\kappa x$ by defining $f(z) = \bigcup\{f(y)\colon y \subseteq z, y \text{ finite}\}$ for infinite $z$. Let $B = \{y\colon y = f(y)\}$. We leave it to the reader to verify that $B$ is as desired.

**2.5. Theorem.** *The functions $\mathcal{M}^\beta$ defined in Definition 2.3 are well-defined.*

**Proof.** By Theorem 2.2(d), infimums of less than $|x|^+$ elements of $\mathscr{B}(\kappa, x)$ always exist, so $\mathcal{M}^1$ is the only possible problem. Suppose $[X] = [Y]$, i.e. $X \cap C = Y \cap C$ for some club $C$. Let $B \subseteq C$ be as in Lemma 2.4 and let $\mathcal{M}(X) = \{y \in X\colon y \cap \kappa$ is an uncountable regular cardinal and $X \cap \mathscr{P}_{(y\cap\kappa)}y$ is stationary in $\mathscr{P}_{(y\cap\kappa)}\}$ and similarly for $\mathcal{M}(Y)$. It is now routine to show that $\mathcal{M}(X) \cap B = \mathcal{M}(Y) \cap B$.

**2.6. Theorem.** *Let $S \subseteq \mathscr{P}_\kappa x$ be stationary. Then $T = \{y \in S\colon y \cap \kappa$ is* **not** *an uncountable regular cardinal or $S \cap \mathscr{P}_{(y\cap\kappa)}y$ is* **not** *stationary in $\mathscr{P}_{(y\cap\kappa)}y\}$ is stationary in $\mathscr{P}_\kappa x$.*

**Proof.** Let $C$ be club. Apply Lemma 2.4 $\omega$ times and then intersect to get a club $B \subseteq C$ such that whenever $y \in B$ and $y \cap \kappa$ is an uncountable regular cardinal, $\mathscr{P}_{(y\cap\kappa)}y \cap B$ is club in $\mathscr{P}_{(y\cap\kappa)}y$. Then $S \cap B \neq \emptyset$ and we may as well assume that $y \cap \kappa$ is an uncountable regular cardinal for each $y \in S \cap B$, for otherwise $T \cap B \neq \emptyset$ and we are done. Pick $y \in S \cap B$ such that $y \cap \kappa$ is least possible. Then $\mathscr{P}_{(y\cap\kappa)}y \cap S \cap B = \emptyset$, so since $\mathscr{P}_{(y\cap\kappa)}y \cap B$ is club, $\mathscr{P}_{(y\cap\kappa)}y \cap S$ is not stationary, and $y \in T$.

**2.7. Corollary.** (a) *If* $a \in \mathcal{B}(\kappa, x)$, $a \neq \mathrm{NS}$, *then* $\mathcal{M}^1(a) < a$.

(b) $(\mathcal{M}^\beta([\mathcal{P}_\kappa x])$: $\beta < m_x(\kappa))$ *is a strictly decreasing sequence in* $\mathcal{B}(\kappa, x)$.

We now want to see how the Boolean algebras $\mathcal{B}(\kappa, x)$ and $\mathcal{B}(\kappa, w)$ 'fit together' when $x \subseteq w$.

**2.8. Lemma.** *Let* $x \subseteq w$ *with* $|x| \geqslant \kappa$.

(a) *If* $B$ *is club in* $\mathcal{P}_\kappa x$, *then* $C = \{z \in \mathcal{P}_\kappa w : z \cap x \in B\}$ *is club in* $\mathcal{P}_\kappa w$.

(b) *If* $C$ *is club in* $\mathcal{P}_\kappa w$, *then there is a* $B \subseteq \{z \cap x : z \in C\}$ *such that* $B$ *is club in* $\mathcal{P}_\kappa x$.

**Proof.** (a) is easy. For (b) define $f: \mathcal{P}_\omega x \to C$ by induction on $|y|$ such that $y \subseteq f(y)$, and $y \subseteq z$ implies $f(y) \subseteq f(z)$. Extend $f$ to $\mathcal{P}_\kappa x$ by letting $f(y) = \bigcup \{f(z): z \subseteq y$ and $z$ finite$\}$ and use Theorem 2.2(a) to show that range$(f) \subseteq C$. Then $B = \{y : f(y) \cap x = y\}$ works.

**2.9. Theorem.** *Let* $\kappa > \omega$ *be regular and let* $x \subseteq w$ *with* $|x| \geqslant \kappa$. *Define* $\iota_{xw}: \mathcal{B}(\kappa, x) \to \mathcal{B}(\kappa, w)$ *by* $\iota_{xw}([X]) = [\{z \in \mathcal{P}_\kappa w : z \cap x \in X]$. *Then* $\iota_{xw}$ *is a well-defined Boolean algebra* **mono***morphism which preserves all infimums of size* $\leqslant |x|$ *and commutes with the appropriate Mahlo operations* $\mathcal{M}^\beta$ *for all* $\mathcal{B} < |x|^+$. *Furthermore, if* $|x| = |w|$, *then* $\iota_{xw}$ *is an isomorphism.*

**Proof.** A routine application of Lemma 2.8.

**2.10. Corollary.** *If* $\kappa \leqslant |x|$, $x \subseteq w$, *and* $m_x(\kappa) < |x|^+$, *then* $m_w(\kappa) = m_x(\kappa)$.

From Corollary 2.10 it would appear that a reasonable way to extend the definition of $m$ is by letting $m^*(\kappa) = \sup_{\lambda \geqslant \kappa} m_\lambda(\kappa)$, but as we will see below, $m_x(\kappa) = |x|^+$ iff $m_w(\kappa) = |w|^+$ for *all* $x$ and $w$ (with $|x|$, $|w| \geqslant \kappa$), so in fact this definition will give nothing new. Before showing this we introduce a different way of looking at the Mahlo operations that will shorten many of the proofs which appear later on.

**2.11. Definition.** Let $\kappa > \omega$ be regular and let $|x| \geqslant \kappa$. If $f$ and $g$ are ordinal-valued functions with domain $\mathcal{P}_\kappa x$, then define the relations $f \sim g$, $f \leqslant g$, and $f < g$ by

$$f \sim g \quad \text{iff} \quad \{y \in \mathcal{P}_\kappa x : f(y) = g(y)\} \text{ contains a club set}$$

and similarly for $\leqslant$ and $<$ (with $=$ replaced by $\leqslant$ and $<$). It is easy to check that $\sim$ is an equivalence relation, $\leqslant$ is transitive and reflexive, and that $<$ is transitive and well-founded. For each $f$, let $\|f\|$ be the rank of $f$ with respect to $<$. Call $f$ *canonical* iff for every $g$, $\|f\| \leqslant \|g\|$ implies $f \leqslant g$.

**2.12. Theorem.** (a) *If $f \leq g$ and $g \leq f$, then $f \sim g$.*

(b) *If $f$ and $g$ are both canonical and $\|f\| = \|g\|$, then $f \sim g$.*

(c) *If $f$ is canonical and $g(y) = f(y) + 1$, then $g$ is canonical and $\|g\| = \|f\| + 1$.*

(d) *If $(f_\gamma : \gamma \in x)$ is a sequence of canonical functions and $f: \mathcal{P}_\kappa x \to \mathrm{ON}$ is defined by $f(y) = \sup_{\gamma \in y} f_\gamma(y)$, then $f$ is canonical and $\|f\| = \sup_{\gamma \in x} \|f_\gamma\|$.*

(e) *There is a sequence $(f_\beta : \beta < |x|^+)$ such that for each $\beta$, $f_\beta$ is canonical and $\|f_\beta\| = \beta$, and if $y \in \mathcal{P}_\kappa x$ is such that $y \cap \kappa$ is regular uncountable, then $f_\beta \restriction \mathcal{P}_{y \cap \kappa} y$ is canonical of rank $f_\beta(y)$.*

(f) *If $w \subseteq x$ and $f$ is defined by $f(y) = \overline{y \cap w}$, then $f$ is canonical and $\|f\| = \bar{w}$, where $\bar{z}$ is the order type of $z$ for $z$ a set of ordinals.*

(g) *For all $\beta < |x|^+$*

$$\mathcal{M}^\beta([\mathcal{P}_\kappa x]) = [\{y \in \mathcal{P}_\kappa x : f_\beta(y) \leq m_y(\overline{y \cap \kappa})\}].$$

**Proof.** (a), (b) and (c) are easy. For (d), let $\beta = \sup_{\gamma \in x} \|f_\gamma\|$. For each $\gamma \in x$, $f_\gamma \leq f$ on the club set $\{y : \gamma \in y\}$, so $\|f\| \geq \beta$. Suppose $\|g\| \geq \beta$. Then $\|g\| \geq \|f_\gamma\|$ for each $\gamma$, so $g \geq f_\gamma$ by canonicity of $f_\gamma$. Pick $C_\gamma$ so that $g \geq f_\gamma$ on $C_\gamma$ and it is routine to show that $g \geq f_\gamma$ on $\Delta(C_\gamma : \gamma \in x)$. For (e) use (c) at successors and (d) at limits and then the second half of (e) is routine to prove by induction. For (f) show by induction on $\bar{w}$ that $f \sim f_{\bar{w}}$, where $f_{\bar{w}}$ is as in (e). To prove (g), proceed by induction on $\beta$. The limit case is easy and using the second half of (e) makes the successor case routine.

**2.13. Theorem.** *If $\kappa > \omega$ is regular and $|x| \geq \kappa$, then $m_x(\kappa) = |x|^+$ iff $m(\kappa) = \kappa^+$ (i.e. iff $\kappa$ is greatly Mahlo).*

**Proof.** If $m(\kappa) \neq \kappa^+$, then $m(\kappa) < \kappa^+$, and $m_x(\kappa) = m(\kappa) < |x|^+$ by Corollary 2.10. Suppose $m_x(\kappa) \neq |x|^+$. Let $\beta = m_x(\kappa)$. Then $|x \cup \beta| = |x|$, so if we let $w = x \cup \beta$, then $m_w(\kappa) = m_x(\kappa)$ by Corollary 2.10. Since $m_w(\kappa) = \beta$, $\mathcal{M}^\beta([\mathcal{P}_\kappa w]) = \mathrm{NS}(\kappa, w)$, i.e. $\{y \subset \mathcal{P}_\kappa w : f_\beta(y) \leq m_y(\overline{y \cap \kappa})\}$ is nonstationary by Theorem 2.12(g), where $f_\beta(y)$ is canonical and $\|f_\beta\| = \beta$. Noting that we can let $f_\beta(y) = \overline{y \cap \beta}$, let

$$C \subseteq \{y \in \mathcal{P}_\kappa w : \overline{y \cap \beta} > m_y(\overline{y \cap \kappa})\}$$

be club. Define $\{y_\alpha : \alpha < \kappa\}$ by induction on $\alpha < \kappa$ such that $y_\alpha \in C$ and $\alpha < \gamma < \kappa$ implies $a \subseteq y_\alpha \subseteq y_\gamma$. Let $t = \bigcup_{\alpha < \kappa} y_\alpha$ and note that $|t| = \kappa$. Then $B = \{y_\alpha : \alpha < \kappa\}$ is club in $\mathcal{P}_\kappa t$, so

$$\{y \in \mathcal{P}_\kappa t : \overline{y \cap \beta} \leq m_y(\overline{y \cap \kappa})\} \cap B = \emptyset.$$

But $f(y) = \overline{y \cap \beta}$ is canonical of rank $\overline{t \cap \beta} < \kappa^+$ in $\mathcal{P}_\kappa t$, so $m(\kappa) = m_t(\kappa) \leq \overline{t \cap \beta} < \kappa^+$.

**2.14. Theorem.** *If $\kappa$ is greatly Mahlo, then for every cardinal $\lambda \geq \kappa$, the nonstationary ideal over $\mathcal{P}_\kappa \lambda$ is not $\lambda^+$-saturated.*

**Proof.** Immediate from Theorem 2.13 and Corollary 2.7.

Theorem 2.13 gives us the fact that $m_x(\kappa)$ cannot take on values in the interval $[\kappa^+, |x|^+)$, so that trying to define new types of Mahlo cardinals using this generalization of the Mahlo operation $\mathcal{M}$ will not work. There is a slight adjustment in the definition, however, that will give the type of cardinals desired. Suppose for a moment that the conclusion of Theorem 2.12(g) were actually the definition of the operation $\mathcal{M}^\beta$ ($m_y(\alpha)$ being defined by induction on $\alpha$). Then we can alter this 'definition' slightly be adding an additional condition. This condition has the advantage of being natural and of causing the analogue of Theorem 2.13 to fail. The function $\bar{m}$ defined below is the same as the function $m^*$ which was defined in [1], and has all of the desirable properties mentioned in Section 1. However, the desire to generalize even further has led to a change in notation which was crucial to make the further generalization readable. We will then get an $m^*$ which is different from $\bar{m}$, but whose relation to $\bar{m}$ can be stated in a precise way. Since the function $\bar{m}$ is for illustration purposes only and will shortly be abandoned, we define only for $\bar{m}(\kappa) \leq \kappa^{+++}$ and leave the obvious induction to further successors of $\kappa$ to the reader.

**2.15. Definition.** Define a function $\bar{m}: \mathrm{ON} \to \mathrm{ON}$ and sets $(M^\kappa_\lambda : \lambda < \kappa^{+++})$ by induction on $\kappa$. Suppose $\bar{m} \restriction \kappa$ has been defined. If $\kappa$ is not regular, let $\bar{m}(\kappa) = 0$ and $M^\kappa_\lambda = \emptyset$ for all $\lambda$. If $\kappa$ is regular and $\lambda < \kappa^{+++}$, let $\mu = |\kappa \cup \lambda|$ and pick $f_\lambda : \mathcal{P}_\kappa\mu \to \mathrm{ON}$ canonical of rank $\lambda$.

*Case* I: $\lambda < \kappa^+$. Let

$$M^\kappa_\lambda = \{y \in \mathcal{P}_\kappa\mu : f_\lambda(y) \leq \bar{m}(\overline{y \cap \kappa})\}.$$

*Case* II: $|\lambda| = \kappa^+$. Let

$$M^\kappa_\lambda = \{y \in \mathcal{P}_\kappa\mu : f_\lambda(y) \leq \bar{m}(\overline{y \cap \kappa}) \text{ and } \bar{y} = |y \cap \kappa|^+\}.$$

*Case* III: $|\lambda| = \kappa^{++}$. Let

$$M^\kappa_\lambda = \{y \in \mathcal{P}_\kappa\mu : f_\lambda(y) \leq \bar{m}(\overline{y \cap \kappa}) \text{ and } \bar{y} = |y \cap \kappa|^{++} \text{ and } \overline{y \cap \kappa^+} = |y \cap \kappa|^+\}.$$

$\bar{m}(\kappa)$ is then defined as the least $\lambda$ such that $M^\kappa_\lambda$ is not stationary, with $\bar{m}(\kappa) = \kappa^{+++}$ if all $M^\kappa_\lambda$'s are stationary.

Note that although the sets $M^\kappa_\lambda$ depend on the choice of $f_\lambda$, their equivalence class modulo the nonstationary ideal is independent of that choice. With this definition it is easy to show that if $\kappa$ is supercompact, then $\bar{m}(\kappa) = \kappa^{+++}$ and that the other properties mentioned as desirable in Section 1 hold.

At present, the consistency strength of these cardinals is not known. If we consider the simplest nontrivial case, $\bar{m}(\kappa) > \kappa^+$, then we know at the upper end that if $\kappa$ is $\kappa^+$-supercompact, then $\bar{m}(\kappa) > \kappa^+$. At the lower end, J. Baumgartner has pointed out that if $\{x \in \mathcal{P}_\kappa\lambda : |x| > |x \cap \kappa|\}$ is stationary, then $0^{\#}$ exists, and it follows immediately from Definition 2.15 that if $\bar{m}(\kappa)>\kappa^+$ for some $\kappa$, then $0^{\#}$ exists.

A similar argument gives that if $\kappa$ is measurable and $\bar{m}(\kappa)>\kappa^+$, then $0^\dagger$ exists, so even a measurable can have $\bar{m}(\kappa)=\kappa^+$. There is some reason to believe that the consistency strength is somewhat lower, however, for we have been able to show that if $\kappa$ is weakly inaccessible and $\lambda>\kappa$ is Ramsey, then $\{x\in P_\kappa\lambda : |x|> |x\cap\kappa|\}$ is stationary. (See [2].) In light of this we consider the following conjecture to be reasonable.

**2.16. Conjecture.** Con(ZFC + $\exists$ Ramsey cardinal) implies

$$\mathrm{Con}(\mathrm{ZFC}+\exists\kappa(\bar{m}(\kappa)>\kappa^+)).$$

If this conjecture turns out to be true, then we will have a relatively narrow range for the consistency strength of these generalized Mahlo cardinals. However, caution is dictated by the fact that we cannot replace 'Con' by direct implication (as was conjectured in the earlier version of this paper), a simple counterexample being $L[\mathcal{U}]$ (see [2] for a proof of this).

## 3. The generalized Mahlo hierarchy

When it comes to carrying the ideas of Definition 2.15 further, there is more than one way to proceed. As pointed out by several people, one way to proceed is to define the $M^\kappa_\lambda$ of Definition 2.15 by

$$\begin{aligned} M^\kappa_\lambda=\{x\in\mathcal{P}_\kappa\,|\kappa\cup\lambda| : f_\lambda(y)\leq\bar{m}(\overline{(y\cap\kappa)}) \text{ and } \forall\gamma\in x\cup\{|\kappa\cup\lambda|\}, \\ \gamma \text{ is a cardinal iff } \overline{x\cap\gamma} \text{ is a cardinal}\}\end{aligned}$$

for all ordinals $\lambda$ and letting $\bar{m}(\kappa)=\infty$ if $M^\kappa_\lambda$ is stationary for all $\lambda$. However, this definition has a property reminiscent of Theorem 2.13, namely that if $\bar{m}(\kappa)$ is greater than the next weakly-Mahlo cardinal, then $\bar{m}(\kappa)=\infty$. One could then add the requirement "$\gamma$ is weakly-Mahlo iff $\overline{x\cap\gamma}$ is weakly-Mahlo" to get further generalizations.

R. Solovay has pointed out another way of generalizing this idea that is different from the definitions of this paper. Using sets of the form $\mathcal{P}_\kappa(V_\lambda)$ instead of $\mathcal{P}_\kappa\lambda$, he defines a function $m'$ such that $m'(\kappa)=\infty$ iff $\kappa$ is supercompact.

The idea we are going to use is to replace "$\gamma$ is a cardinal iff $\overline{x\cap\gamma}$ is a cardinal" in the above definition by $m(\gamma)=m(\overline{x\cap\gamma})$, or at least by something close enough to that idea to make everything well-defined. In order to make things go more smoothly, we wish to make a couple of modifications. We first note that it is somewhat annoying to have to pick a canonical function $f_\lambda$ of rank $\lambda$ at each state. Note, however, that if we use $\mathcal{P}_\kappa\lambda$ instead of $\mathcal{P}_\kappa\,|\kappa\cup\lambda|$ at each state, then $f_\lambda(x)=\bar{x}$ is canonical of rank $\lambda$, so no choosing needs to be done. This causes problems if $\lambda<\kappa$, but it is easily solved by using $\kappa$ instead of 0 as our starting point, so in our definition of $m^*$, we will always have $m^*(\kappa) \geqslant \kappa$, and we will have $m^*(\kappa) \geqslant \kappa+1$ iff $\kappa$ is uncountable and regular, $m^*(\kappa) \geqslant \kappa+2$ iff $\kappa$ is weakly-Mahlo, and so forth. This 'shift by $\kappa$' will disappear when $\lambda \geqslant \kappa \cdot \omega$, long before anything really interesting happens.

The new definition also allows many more possibilities for the exact value of $m^*(\kappa)$, such as

$$m^*(\kappa) = (\text{the next greatly-Mahlo})^2 + \kappa^{+++} + 36,$$

for example, and many terms of this type.

**3.1. Definition.** Let $<_{\mathrm{m}}$ be the reverse lexicographical ordering on ordered pairs of ordinals.

The above ordering seems to be the most natural one for doing the following definition. However, different orderings might give analogous definitions that are also interesting.

To avoid dealing with special cases, we adopt the convention from this point on that if $\kappa$ is not an uncountable regular cardinal, then *every* subset of $\mathscr{P}_\kappa\lambda$ is nonstationary.

**3.2. Definition.** For each $(\alpha, \beta)$ with $\alpha \leqslant \beta$ define sets $Q^\alpha_\beta \subseteq \mathscr{P}_\alpha\beta$ by induction on $<_{\mathrm{m}}$:

Let $\kappa \leqslant \lambda$ and suppose $Q^\alpha_\beta$ has been defined for all $(\alpha, \beta) <_m (\kappa, \lambda)$. Let

$$\begin{aligned} Q^\kappa_\lambda = \{x \in \mathscr{P}_\kappa\lambda \colon &\text{(a) } Q^{\overline{\kappa \cap x}}_{\bar{x}} \text{ is nonstationary.} \\ &\text{(b) } (\forall \gamma \in x) Q^{\overline{\kappa \cap x}}_{\overline{\gamma \cap x}} \text{ is stationary.} \\ &\text{(c) If } \gamma, \delta \in x \text{ and } \gamma \leqslant \delta, \text{ then } Q^\gamma_\delta \text{ is stationary} \\ &\quad \text{iff } Q^{\overline{\gamma \cap x}}_{\overline{\delta \cap x}} \text{ is stationary}\}. \end{aligned}$$

$m^*(\kappa)$ is defined to be the least $\lambda$ such that $Q^\kappa_\lambda$ is nonstationary. If $Q^\kappa_\lambda$ is stationary for all $\lambda \geqslant \kappa$, we define $m^*(\kappa) = \infty$. Note that (a) and (b) together say that $m^*(\overline{\kappa \cap x}) = \bar{x}$. Intuitively, (c) is intended to say that $m^*(\overline{\gamma \cap x}) = \overline{m^*(\gamma) \cap x}$, or at least as close as we can come to that considering the order in which the $Q^\kappa_\lambda$'s were defined.

**3.3. Theorem.** *If $\alpha < \kappa$ and $m^*(\alpha) \geqslant \kappa$, then $m^*(\alpha) \geqslant m^*(\kappa)$.*

**Proof.** Let $\beta = m^*(\alpha)$, and suppose $Q^\kappa_\beta$ is stationary. $Q^\alpha_\beta$ is not stationary, so pick a club $C \subseteq \mathscr{P}_\alpha\beta$ such that $C \cap Q^\alpha_\beta = 0$. Let

$$\begin{aligned} B = \{ \in \mathscr{P}_\kappa\beta : &\, x \cap \kappa \text{ is an ordinal, } \alpha \subseteq x, \text{ and} \\ &\mathscr{P}_\alpha x \cap C \text{ is club in } \mathscr{P}_\alpha x\}. \end{aligned}$$

Then $B$ is club in $\mathscr{P}_k\beta$, so let $x \in B \cap Q^\kappa_\beta$. Let $f : \bar{x} \to x$ be the order preserving map and suppose $y \in Q^\alpha_{\bar{x}}$. Let $z = f''(y)$. It is now easy to check (using the fact that

$x \in Q^{\kappa}_{\beta}$ and $y \in Q^{\alpha}_{\bar{x}}$) that $z \in Q^{\alpha}_{\beta}$. Thus $\{f''(y) : y \in Q^{\alpha}_{\bar{x}}\} \cap (C \cap Q^{\alpha}_{x}) = \emptyset$, so $Q^{\alpha}_{x}$ is not stationary, i.e., $m^*(\alpha) \leqslant \bar{x} < \kappa$.

**3.4. Corollary.** *$m^*(\kappa)$ is never an uncountable regular cardinal.*

**Proof.** If $\kappa$ is not regular, then $m^*(\kappa) = \kappa$. If $\kappa$ is regular, then $m^*(\kappa) > \kappa$ and if $\lambda > \kappa$ is regular with $m^*(\kappa) \geqslant \lambda$, then by the theorem $m^*(\kappa) \geqslant m^*(\lambda) > \lambda$.

Theorem 3.3 may seem at first glance to be a rather undesirable property for $m^*$ to have but there is an analogy from a different large cardinal property: For each ordinal $\alpha$, let

$$s(\alpha) = \begin{cases} \text{the least } \lambda \geqslant \alpha \text{ such that } \alpha \text{ is not } \lambda\text{-supercompact}, \\ \infty \quad \text{if } \alpha \text{ is supercompact}. \end{cases}$$

Then Theorem 3.3 is true with $m^*$ replaced by $s$, and is in fact merely a statement in different notation of a well-known result (see [10]).

The following theorem indicates that Theorem 3.3 is the only major obstacle to the desired properties listed in Section 1.

**3.5. Theorem.** (a) *If $\alpha \leqslant \beta < \gamma$ and $Q^{\alpha}_{\beta}$ is nonstationary, then $Q^{\alpha}_{\gamma}$ is nonstationary.*

(b) *Let $\bar{m}$ be as in Definition* 2.15. *Then $\bar{m}$ is completely determined from $m^*$:*

*Case* I: $m^*(\alpha) < \alpha^+$. *Then $m^*(\alpha) = \alpha + \bar{m}(\alpha) = \alpha + m(\alpha)$.*

*Case* II: $\alpha^+ < m^*(\alpha) < \alpha^+ + \omega$. *Then $m^*(\alpha) = \bar{m}(\alpha) + 1$.*

*Case* III: $\alpha^+ + \omega \leqslant m^*(\alpha) < \alpha^{++}$. *Then $m^*(\alpha) = \bar{m}(\alpha)$.*

*Case* IV: $\alpha^{++} < m^*(\alpha) < \alpha^{++} + \omega$. *Then $m^*(\alpha) = \bar{m}(\alpha) + 1$.*

*Case* V: $\alpha^{++} + \omega \leqslant m^*(\alpha) < \alpha^{+++}$. *Then $m^*(\alpha) = \bar{m}(\alpha)$.*

*Case* VI: $m^*(\alpha) > \alpha^{+++}$. *Then $\bar{m}(\alpha) = \alpha^{+++}$, with the obvious generalization if the definition of $\bar{m}$ is carried further than $\alpha^{+++}$.*

(c) *If $m^*(\alpha) > \alpha^+\alpha^+ + n$, where $1 \leqslant n < \omega$, then $\{\gamma < \alpha : m^*(\gamma) = \gamma^+ + n\}$ is stationary in $\alpha$.*

(d) *If $\kappa$ is supercompact, then $m^*(\kappa) = \infty$.*

**Proof.** (a) If $Q^{\alpha}_{\beta}$ is nonstationary, then let $C = \{x \in \mathcal{P}_{\alpha}\gamma : \{\alpha, \beta\} \subseteq x\}$. Then $Q^{\alpha}_{\gamma} \cap C = \emptyset$, for if $x \in Q^{\alpha}_{\gamma} \cap C$, then $Q^{\overline{\alpha \cap x}}_{\overline{\beta \cap x}}$ is nonstationary by 3.2(c), contradicting 3.2(b).

(b) If $\alpha$ is a cardinal, let $g : \alpha^{+++} \to \alpha^{+++} - (\alpha \cup \{\alpha^+\} \cup \{\alpha^{++}\})$ be the order preserving map. Let $M^{\kappa}_{\lambda}$ be as in Definition 2.15. Then it is easy to prove by induction on $\kappa$ that $\iota_{|\kappa \cup \lambda|, \lambda}[M^{\kappa}_{\lambda} - M^{\kappa}_{\lambda+1}] = [Q^{\kappa}_{g(\lambda)}]$. The rest then follows easily since $\iota_{|\kappa \cup \lambda|, \lambda}$ is an isomorphism.

(c) Intersect the club set $C = \{x \in \mathcal{P}_{\alpha}(\alpha^+ + n) : \alpha^+ \leqslant \gamma < \alpha^+ + n \text{ implies } \gamma \in x\}$ with $Q^{\alpha}_{\alpha^+ + n}$. The rest is then easy.

(d) uses the usual argument. If $m^*(\kappa) = \lambda < \infty$, then the elementary embedding obtained from a normal ultrafilter over $\mathcal{P}_\kappa\lambda$ can be used to get the usual contradiction.

The result of 3.5(c) is typical of what can be proven using more complicated terms. In fact Theorem 3.5(c) is true of any 'reasonable' term $\tau(\alpha)$ replacing $\alpha^+ + n$ which does not collide with the result of Theorem 3.3. Thus, 3.5(c) cannot be proven for $\tau(\alpha) = $(the least weakly Mahlo$>\alpha)+1$, but is true for $\tau(\alpha) = $(the least weakly Mahlo$>\alpha)+2$, using the club set $C = \{x \in \mathcal{P}_\alpha\beta : \{\delta, \delta+1\} \subseteq x\}$, where $\delta$ is the least weakly Mahlo$>\alpha$ and $\beta = \delta + 2$.

Before proceeding with applications we note one other generalization. As is well known, $\kappa$ is called *strongly-Mahlo* iff the strongly inaccessible cardinals below $\kappa$ form a stationary set. All of the results in this paper have a corresponding result which generalizes the strong Mahlo hierarchy and which in fact can be obtained by systematically replacing the word 'regular' by the words 'strongly inaccessible' in every definition and theorem in this paper. (Practically any other large cardinal property could also be switched with 'regular' in this paper, but 'regular' and 'strongly inaccessible' seem more interesting than any others in that they generalize previous definitions.)

## 4. Canonical filter sequences

Let $\kappa$ be a measurable cardinal and let $\mathcal{U}$ be a normal ultrafilter over $\kappa$. Then, by a well known result of Kunen, for any regular $\rho > 2^\kappa$ there is an elementary embedding $j : L[\mathcal{U}] \to L[C]$, where $C$ is the club filter over $\rho$. Thus the club filters $C$ can be considered as a canonical collection which provides 'all' models L$[\mathcal{U}]$ (in the sense that every such model is an elementary substructure of such an $L[C]$). The main result of this section is to provide canonical filter sequences which play the same role for the models $L[U]$ with many measurable cardinals that were defined by Mitchell in [8] and [9].

Because the definitions are substantially easier, we first give the definitions and the canonical form theorem for the models of [8].

**4.1. Definition** (Mitchell). $U$ is called a *coherent sequence* iff $U$ is a function with dom $U = \{(\alpha, \beta) : \beta < O^U(\alpha)$ and $\alpha < l^U\}$ for some ordinal $l^U$ and ordinal-valued function $O^U$, each $U(\alpha, \beta)$ is a filter over $\alpha$, and the following properties hold:

(a) For each $(\alpha, \beta) \in \text{dom } U$, $U(\alpha, \beta) \cap L[U]$ is a normal ultrafilter in $L[U]$.

(b) If $i : L[U] \to L[U]^\alpha / U(\alpha, \beta)$ is the canonical embedding, then $O^{i(U)}(\alpha) = \beta$ and for every $(\alpha', \beta') \in \text{dom } U$ which is lexicographically less than $(\alpha, \beta)$, $(i(U))(\alpha', \beta') = U(\alpha', \beta')$.

See [8] for more details on the properties of these models $L[U]$.

**4.2. Definition** (The canonical filter sequences). For each ordinal $\alpha$ and each $\beta < m(\alpha)$, let $\mathcal{F}(\alpha, \beta)$ be the filter generated by the element $a =$

$\mathcal{M}^\beta([\alpha])-\mathcal{M}^{\beta+1}([\alpha])$, i.e. $\mathcal{F}(\alpha,\beta)=\{X\subseteq\alpha:[X]\geqslant a\}$. If $[\mu,\nu)$ is any interval of ordinals, let $\mathcal{F}_{[\mu,\nu)}=\mathcal{F}\restriction\{(\alpha,\beta):\mu\leqslant\alpha<\nu$ and $\beta<m(\alpha)\}$.

**4.3. Theorem.** *Suppose there is a proper class of greatly Mahlo cardinals. Then for any coherent sequence $U$ there is an interval $[\mu,\nu)$ and an elementary embedding $i:L[U]\to L[\mathcal{F}_{[\mu,\nu)}]$.*

A hypothesis of the type appearing in the first sentence of this theorem cannot be avoided. For example, we could have a model of ZFC having a measurable limit of measurable cardinals with no larger weakly inaccessible cardinal. The theorem would be false in such a model without the first sentence.

We wish to avoid writing out the proof of Theorem 4.3 in complete detail, as it follows from the much stronger Theorem 4.8. On the other hand, the proof of Theorem 4.8 is several orders of magnitude more difficult. Thus, we will outline the main arguments for the special case of Theorem 4.3, and the reader is advised to look at this case first before trying to work out all of the details of the most general case. A few definitions and theorems used in this outline do not appear until later in this paper.

Let $U$ be coherent, let $\theta$ be least such that $i_{\mathcal{U}}\theta=\theta$ for all $\mathcal{U}\in\mathrm{range}(U)$, and note that for any regular $\kappa>\theta$ it requires $\kappa$ iterated ultrapowers of $L[U]$ to move $\theta$ above $\kappa$. Let $\mu>\theta$ be regular but not weakly-Mahlo (i.e. $m(\mu)=1$) and define $(M_\alpha:\alpha\leqslant\mu)$ by iterating the smallest measure of $M_0=L[U]$ $\mu$ times. If $M_\alpha$ has been defined for some $\alpha\geqslant\mu$ let $\kappa_\alpha$ be the least $\kappa\geqslant\mu$ such that $(i_\alpha O^U)(\kappa)>m(\kappa)$, let $\mathcal{U}_\alpha=(i_\alpha U)(\kappa_\alpha,m(\kappa_\alpha))$, and let $M_{\alpha+1}=M_\alpha^{\kappa_\alpha}/\mathcal{U}_\alpha$, and proceed until such time (if any) that no $\kappa_\alpha$ as defined exists.

Because this case is simpler we can consider only the $M_\alpha$'s and ignore the $M_x$'s defined in Definition 4.9, and we can replace $\mathcal{P}_\kappa\kappa$ by $\kappa$ and ignore all $\mathcal{P}_\kappa x$'s. This makes the proof of the analogue of Corollary 4.13 a little more involved (since the 'nice' canonical functions $f_\beta(x)=\overline{x\cap\beta}$ are often not available) but still routine, while Theorem 4.14 is easier because the $\kappa_\alpha$'s are strictly increasing. Whether the indexing of filters over $\kappa$ starts at 0 or at $\kappa$ is irrelevant.

Let $\kappa\geqslant\mu$ be regular such that $M_\kappa$ is defined and let $\lambda<i_\kappa\theta$. Let

$$A(\kappa,\lambda)=\{\alpha<\kappa:\kappa_\alpha=\alpha,\ i_{\alpha\kappa}\kappa_\alpha=\kappa,\text{ and } i_{\alpha\kappa}(m(\alpha))=\lambda\}.$$

Thus by Corollary 4.13 (since $\kappa_\alpha=\alpha$ and $i_{\alpha\kappa}\kappa_\alpha=\kappa$ happen on a club) there is an $f_\lambda$ canonical of rank $\lambda$ such that $A(\kappa,\lambda)=\{\alpha<\kappa:f_\lambda(\alpha)=m(\alpha)\}$, so $[A]=\mathcal{M}^\lambda([\kappa])-\mathcal{M}^{\lambda+1}([\kappa])$ by the analogue for $\kappa$ of Theorem 2.12(g). Thus if $\lambda<m(\kappa)$, then $A(\kappa,\lambda)$ is stationary and $i_{\alpha\kappa}\mathcal{U}_\alpha=(i_\kappa U)(\kappa,\lambda)$ for each $\alpha\in A(\kappa,\lambda)$. Thus, by the usual arguments, if $X\in\mathcal{P}(\kappa)\cap M_\kappa$, then $X\in(i_\kappa U)(\kappa,\lambda)$ iff $X$ contains a final segment of $A(\kappa,\lambda)$, i.e. iff $[X]\geqslant\mathcal{M}^\lambda([\kappa])-\mathcal{M}^{\lambda+1}([\kappa])$, since $A(\kappa,\lambda)$ is stationary. Future iterated ultrapowers do not change this. Further, since the $A(\kappa,\lambda)$'s are pairwise disjoint, only $\kappa$ many of them can be nonempty, so $m(\kappa)<\kappa^+$. Thus the iterations $M_\alpha$ must stop before reaching the first greatly Mahlo cardinal greater than $\mu$. Letting $\nu=\{\alpha:M_\alpha$ is defined$\}$, it is routine to check that $\mathscr{F}_{[\mu,\nu)}$ is the desired canonical filter sequence.

(*Note.* As a second intermediate step in attacking this proof the reader might wish to consider only sequences satisfying Definition 4.5 [where the $\kappa_\alpha$'s are still strictly increasing] before jumping into the much more general sequences of Definition 4.6.)

We repeat some definitions from [9]:

**4.4. Definition.** If $f$ is a function with domain ${}^\delta\alpha$, and $x\subseteq\delta$, $f$ is said to have *support in* $x$ if there is a function g with domain ${}^x\alpha$ such that for all $a\in{}^\delta\alpha$, $f(a)=g(a\restriction x)$. The unique smallest $x$ such that $f$ has support in $x$ will be called Supp($f$). The same definition applies to subsets of ${}^\delta\alpha$ by referring to the characteristic function. Let

$$\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)=\{X\in\mathscr{P}({}^\delta\alpha):\mathrm{Supp}(X)\text{ is finite}\}.$$

$\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$ is a subalgebra of the Boolean algebra $\mathscr{P}({}^\delta\alpha)$, and if $\mathscr{U}\subseteq\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$ is an ultrafilter, then ultrapowers can be formed from $\mathscr{U}$ by using only functions with finite support. Such a $\mathscr{U}$ is called countably complete if $\bigcap A\neq\emptyset$ for all countable $A\subseteq\mathscr{U}$ (note that $\bigcap A\in\mathscr{U}$ is not required and will not be true in general if $\delta$ is infinite). We will abuse notation by identifying $X\subseteq{}^\delta\alpha$ with $\{a\restriction x:a\in X\}$ for any $x$ such that $\mathrm{Supp}(X)\subseteq x\subseteq\delta$ (and similarly for functions). We can then consider $\mathscr{P}_{\mathrm{f}}({}^x\alpha)$ to be a 'subset' of $\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$ when $x\subseteq\delta$. If $\mathscr{U}\subseteq\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$ is a filter, let $\mathscr{U}[x]$ be $\mathscr{U}$'$\cap$'$\mathscr{P}_{\mathrm{f}}({}^x\alpha)$. (Quotes will often be used to warn the reader about these notation abuses.) If $M$ is any model of ZFC and $\mathscr{U}\in\mathcal{M}$ is any such ultrafilter. Ult($M$, $\mathscr{U}$) will denote the ultrapower of $M$ by $\mathscr{U}$ (using only functions in $M$) and $i_{\mathscr{U}}:M\to$ Ult($M$, $\mathscr{U}$) will be the appropriate canonical embedding.

We now use ultrafilters of this type to extend the definition of a coherent sequence. The following definition is the same as that given in [9] by Mitchell, except for the adjustment in the indexing mentioned already in Section 3 (so Mitchell's $U(\alpha,\beta)$ will correspond to our $U(\alpha,\alpha+\beta)$, the distinction disappearing when $\beta\geqslant\alpha\cdot\omega$).

**4.5. Definition.** Let $l^U$ be an ordinal and let $O^U$ be an ordinal-valued function with $O^U(\alpha)\geqslant\alpha$ for all $\alpha$ such that $\alpha<\gamma\leqslant O^U(\alpha)$ implies $O^U(\gamma)=\gamma$. Let $U$ be a function with domain $\{(\alpha,\beta):\alpha<l^U$ and $\alpha\leqslant\beta<O^U(\alpha)\}$ such that for each $(\alpha,\beta)\in\mathrm{dom}(U)$, $\mathscr{U}=U(\alpha,\beta)$ is an ultrafilter on $\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$ for some $\delta=\delta(\alpha,\beta)$ such that

(1) $\forall\lambda<\alpha$, $\{a:\lambda<a_0\}\in\mathscr{U}$, and if $\nu<\mu<\delta$, then $\{a:a_\nu<a_\mu<O^U(a_0)\}\in\mathscr{U}$, where for $a\in\mathscr{P}_{\mathrm{f}}({}^\delta\alpha)$, we let $a=(a_\nu:\nu<\delta)$.

(2) If $\nu<\delta$ and $\{a:f(a)<a_\nu\}\in\mathscr{U}$, then there is an $f'\in L[U\restriction(\alpha,\beta)]$ with support in $\nu$ such that $\{a:f(a)=f'(a)\}\in\mathscr{U}$.

(3) If $f\in L[U\restriction(\alpha,\beta)]$ has support in $\nu$, then $\{a:f(a)=a_\nu\}\notin\mathscr{U}$.

(4) $\mathcal{U}$ is countably complete.

(5) $i_{\mathcal{U}}(U) \upharpoonright \alpha+1$ '=' $U \upharpoonright (\alpha, \beta)$, $(\alpha, \beta) \notin \operatorname{dom} i_{\mathcal{U}}(U)$, and if $\{a : f(a) < O^U(a_0)\} \in \mathcal{U}$, then there is an $f' \in L[U \upharpoonright (\alpha, \beta)]$ such that $\{a : f(a) = f'(a)\} \in \mathcal{U}$.

Here we define $F \upharpoonright (\alpha, \beta) = F \upharpoonright \{(\alpha', \beta') \in \operatorname{dom}(F) : \alpha' < \alpha'$ or $\alpha' = \alpha$ and $\beta < \beta\}$ and $F \upharpoonright \alpha+1 = F \upharpoonright (\alpha, O^F(\alpha))$.

For the many nice properties of these models $L[U]$, such as GCH, see [9]. (We will not need these properties in this paper.)

In order to apply the generalized Mahlo hierarchy to these canonical forms, we wish to generalize Definition 4.5 even further. In particular, since $\gamma$ is measurable in $L[U]$ iff $O^U(\gamma) > \gamma$ (corresponding to $O^U(\gamma) > 0$ in [8]) we wish to have cardinals $\gamma$ with $O^U(\gamma) > \gamma$ and $O^U(\alpha \geqslant \gamma$ for some $\alpha < \gamma$. We therefore make the following modifications:

**4.6. Definition.** Extend Definition 4.5 to more sequences $U$ by making the following two changes.

(a) Replace the requirement that $\alpha < \gamma \leqslant O^U(\alpha)$ implies $O^U(\gamma) = \gamma$ by the requirement that if $O^U(\beta) > \beta$, then $\{\alpha < \beta : O^U(\alpha) > \beta\}$ is bounded below $\beta$. (This requirement can be eliminated entirely by more complicated arguments.)

(b) Redefine $F \upharpoonright (\alpha, \beta)$ as $F \upharpoonright \{(\alpha', \beta') : (\alpha', \beta') <_{\mathrm{m}} (\alpha, \beta)\}$ where $<_{\mathrm{m}}$ is the reverse lexicographic order and $F$ is any sequence.

It is easy to check that every $U$ satisfying Definition 4.5 also satisfies Definition 4.6. We note that this definition satisfies the following three properties:

(1) If $\kappa < \lambda \leqslant O^U(\kappa)$, then $O^U(\lambda) \leqslant O^U(\kappa)$ (otherwise let $\mathcal{U} = U(\lambda, O^U(\kappa))$ and the coherence property (5) is violated at the pair $(\kappa, O^U(\kappa))$ for $i_{\mathcal{U}}$).

(2) Given appropriate large cardinal assumptions (e.g. a supercompact cardinal), sequences $U$ exist satisfying Definition 4.6 but not Definition 4.5.

(3) The comparison lemma, (corresponding to Lemma 2.3 in [8] or Lemma 2.3 in [9]), the crucial tool for studying the structure of these models, can be proven (it is immediate from Theorem 4.8).

We now give the definition of the canonical filter sequences that generalize Definition 4.2. Because of the frequent occurrence of $L[U \upharpoonright (\alpha, \beta)]$ in Definition 4.5 we cannot define a single class function $\mathcal{F}$ as in Definition 4.2. Instead we must define functions $\mathcal{F}_\mu$ for each $\mu$ which is regular but not weakly-Mahlo.

**4.7. Definition.** Let $\mu$ be any ordinal such that $m^*(\mu) = \mu + 1$ (i.e. $m(\mu) = 1$). Let $\operatorname{dom}(\mathcal{F}_\mu) = \{(\alpha, \beta) : \mu \leqslant \alpha$ and $\alpha \leqslant \beta < m^*(\alpha)\}$. We define $\mathcal{F}_\mu(\alpha, \beta)$ by induction on $<_{\mathrm{m}}$, so assume $\mathcal{F}_\mu \upharpoonright (\alpha, \beta)$ has been defined and let $M = L[\mathcal{F}_\mu \upharpoonright (\alpha, \beta)]$. Let $F[1]$ be the filter on $\mathcal{P}_{\mathrm{f}}({}^1\mathcal{P}_\alpha\beta)$ 'generated by' $Q^\alpha_\beta$, i.e. $F[1]$ '=' $\{X \in P_\alpha\beta : [X] \geqslant [Q^\alpha_\beta]\}$ (using the natural identification of $P_\alpha\beta$ and ${}^1\mathcal{P}_\alpha\beta$). Suppose the filter $F[\eta]$ on $\mathcal{P}_{\mathrm{f}}({}^\eta\mathcal{P}_\alpha\beta)$ has been defined. Let $\mathfrak{b}_\eta$ be the least ordinal *not* in the set $\{\gamma < \beta :$ there is a canonical $f$ with $\|f\| = \gamma$ and a $g \in M$ with finite support in

$\eta$ such that $\{a \in {}^{\eta}\mathcal{P}_{\alpha}\beta : f(a_0) = g(a^*)\} \in F[\eta]\}$, where $a^* \in {}^{\eta}\alpha$ is defined by $a^*_\rho = a_\rho \cap \alpha\,(\rho < \eta)$ for $a \in {}^{\eta}\mathcal{P}_{\alpha}\beta$. If $\mathfrak{b}_\eta \geq \beta$, then the induction stops and we let $F = F[\eta]$. If $\mathfrak{b}_\eta < \beta$, let

$$F[\eta+1] = \{X \in \mathcal{P}_{\mathrm{f}}({}^{\eta+1}\mathcal{P}_{\alpha}\beta) : \{a \restriction \eta : a \in X \text{ and } \overline{a_0 \cap \mathfrak{b}_\eta} = \overline{a_\eta \cap \alpha}\} \in F[\eta]\}.$$

If $\eta$ is limit let $F[\eta] = `\bigcup'_{\delta<\eta} F[\delta]$. It is easy to show that the $F[\nu]$'s are proper filters such that $F[\nu]` \subseteq 'F[\nu+1]$ and that the $\mathfrak{b}_\nu$'s are strictly increasing, so the induction stops. Thus $F$ is a well-defined filter on $\mathcal{P}_{\mathrm{f}}({}^{\delta}\mathcal{P}_{\alpha}\beta)$ for some $\delta \geq 1$. We now define $\mathcal{F}_\mu(\alpha, \beta) = \{X^* : X \in F\}$, where $X^* = \{a^* : a \in X\}$ for $X \in \mathcal{P}_{\mathrm{f}}({}^{\delta}\mathcal{P}_{\alpha}\beta)$. Thus each $\mathcal{F}_\mu(\alpha, \beta)$ is a filter on $\mathcal{P}_{\mathrm{f}}({}^{\delta}\alpha)$ for some $\delta$.

We are now in a position to state the main theorem of this section. The remainder of the paper will then build the machinery necessary to prove this theorem. The statement "enough high degree Mahlo cardinals" will be made precise before the Theorem is proven. If we consider only sequences $U$ which satisfy Definition 4.5, then a proper class of greatly Mahlo cardinals is enough.

**4.8. Theorem.** *Assume there are 'enough' high degree Mahlo cardinals. Let $U$ be any coherent sequence as in Definition* 4.6. *Then there are ordinals $\mu$ and $\nu$ and an elementary embedding*

$$j : L[U] \to L[\mathcal{F}_\mu \restriction \nu+1].$$

*Iterated ultrapowers*

**4.9. Definition.** Let $M_0$ a standard model of ZFC. We define an iterated ultrapower $M_\alpha : \alpha \leq \lambda$ to be any member of a sequence defined as follows: If $M_\gamma$ has been defined for $\gamma < \alpha$ with elementary embeddings $i_{\gamma\beta} : M_\gamma \to M_\beta\,(\gamma \leq \beta < \alpha)$ which commute ($i_{\alpha\beta} = i_{\gamma\beta} i_{\delta\gamma}$, $\delta \leq \gamma \leq \beta < \alpha$) and $\alpha$ is a limit ordinal, $M_\alpha$ is the direct limit of $(M_\gamma : \gamma < \alpha)$ with respect to the embeddings $i_{\gamma\beta}$. If $M_\alpha$ has been defined, then $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, \mathcal{U}_\alpha)$ for some $\mathcal{U}_\alpha \in M_\alpha$ such that $M_\alpha \models$"$\mathcal{U}_\alpha$ is a countably complete ultrafilter over some $\mathcal{P}_f({}^{\delta}\kappa)$". We let $i_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ be the canonical embedding and let $\kappa_\alpha$ be the critical point of $i_{\alpha,\alpha+1}$. If $\omega_1 \subseteq M_0$, then all $M_\alpha$'s are well-founded (see [7] on how to prove this) and will be identified with their transitive collapse. From the linear system $(M_\alpha : \alpha < \lambda)$ we want to construct a directed system $(M_x : x \in D)$ for some directed $D \subseteq \mathcal{P}(\lambda)$, with commuting maps $i_{xy}$ for $x, y \in D$, $x \subseteq y$. The definition is by induction:

*Case* I: *Limit case.* Let $\alpha$ be a limit ordinal and suppose $i_{yz}$ and $\mathcal{P}(\beta) \cap D$ have been defined for all $y \subseteq z \subseteq \beta < \alpha$. If $x$ is an unbounded subset of $\alpha$, then we let $x \in D$ iff proper initial segment of $x$ is in $D$ and $M_x$ is the direct limit of $\{M_y : y$ is a proper initial segment of $x\}$.

We now define $i_{wx}$ in the obvious way for $w \subseteq x$ and $x$ unbounded in $\alpha$.

*Case* II: *Successor case*: Suppose $\mathcal{P}(\alpha) \cap D$ and $\{i_{x\alpha} : x \in \mathcal{P}(\alpha) \cap D\}$ have been defined and let $x \subseteq \alpha$. If $x \notin D$, then we will have $x \cup \{\alpha\} \notin D$. If $x \in D$, then we let

$x\cup\{\alpha\}\in D$ iff $\mathcal{U}_\alpha\in\text{range}(i_{x\alpha})$, and if $\mathcal{U}_{x\alpha}$ is such that $i_{x\alpha}\mathcal{U}_{x\alpha}=\mathcal{U}_\alpha$, we let $M_{x\cup\{\alpha\}}=\text{Ult}(M_x,\mathcal{U}_{x\alpha})$. For $x\subseteq y\subseteq\alpha$, $x$, $y\in D$, the map $i_{x\cup\{\alpha\},y\cup\{\alpha\}}$ is the map $[f]_{\mathcal{U}_{x\alpha}}\mapsto[i_{xy}f]_{\mathcal{U}_{y\alpha}}$, for all functions $f\in M_x$ whose domains are appropriate to the ultrafilter $\mathcal{U}_{x\alpha}$. (See [8] for a different, but equivalent, definition of these models $M_x$.) We write $i_x$ for $i_{0x}$. When we say $\underline{M_x \text{ exists}}$ we will mean $x\in D$. Some of the basic properties of this directed system are listed in the following theorem.

**4.10. Theorem.** *Let $(M_x : x\in D)$ be as in Definition 4.9.*

(a) *$D$ is closed under unions (but not necessarily under intersections).*

(b) *If $E\subseteq D$ is directed under inclusion and $x=\bigcup E$, then $M_x$ is the direct limit of $(M_y : y\in E)$.*

(c) *For any $\alpha\leqslant\lambda$ and any $y\subseteq\alpha$ there is an $x\in D$ such that $y\subseteq x\subseteq\alpha$. If $y$ is finite, then we can pick $x$ to be finite. If $y$ is infinite, $x$ can be picked so that $|x|=|y|$.*

(d) *If $x\in D$ and $\forall\gamma\in x$, $i_{x\cap\gamma,\gamma}\mathcal{U}_{\overline{x\cap\gamma}}=\mathcal{U}_\gamma$, then $i_x=i_{\bar{x}}$ and $M_x=M_{\bar{x}}$. (Note that this does not necessarily imply $i_{x\lambda}=i_{\bar{x}\lambda}$.)*

**Outline of proof.** (a) and (b) are routine. For (c), first prove the finite case by induction on $|y|$. The infinite case then follows easily from the finite case and (a). Prove (d) by induction on the order type of $x$.

Our primary interest in these models $M_x$ lies in the following corollary:

**4.11. Corollary.** *If $\kappa$ is regular and uncountable, $\kappa\leqslant\lambda$, then $\{x\in\mathcal{P}_\kappa\lambda : M_x \text{ exists}\}$ is club in $\mathcal{P}_\kappa\lambda$.*

**4.12. Lemma.** *Suppose $\alpha$ is regular, $M_\nu$ exists, $\alpha\leqslant\nu\leqslant\beta$, and $\rho\leqslant\beta$ is such that whenever $y\in\mathcal{P}_\alpha\nu$ and $M_y$ exists, $|\text{range}(i_{y\nu})\cap\rho|<\alpha$. Then*

$$C=\{x\in\mathcal{P}_\alpha\beta : M_{x\cap\nu} \text{ exists}, \{\alpha,\beta,\rho\}\subseteq\text{range}(i_{x\cap\nu,\nu}),\ \text{and } x\cap\rho=\text{range}(i_{x\cap\nu,\nu})\cap\rho\} \text{ is club.}$$

**Proof.** Suppose $B\subseteq C$ is a chain and let $x=\bigcup B$. Then

$$x\cap\rho=\bigcup_{z\in B}(z\cap\rho)=\bigcup_{z\in B}(\text{range}(i_{z\cap\nu,\nu})\cap\rho)$$
$$=\left(\bigcup_{z\in B}\text{range}(i_{z\cap\nu,\nu})\right)\cap\rho=\text{range}(i_{x\cap\nu,\nu})\cap\rho,$$

since $M_{x\cap\nu}$ is the direct limit of the $(M_{z\cap\nu})$'s. $M_{x\cap\nu}$ exists and $\langle\alpha,\beta,\rho\}\subseteq\text{range}(i_{x\cap\nu,\nu})$ are clear, so $x\in C$ and $C$ is closed.

For unboundedness, suppose $z\in\mathcal{P}_\alpha\beta$. Pick $z(0)\supseteq z$ so that $M_{z(0)\cap\nu}$ exists. If $z(n)$ has been defined pick $z(n+1)\in\mathcal{P}_\alpha\beta$ such that

(a) $z(n)\subseteq z(n+1)$ and $M_{z(n+1)\cap\nu}$ exists,

(b) $\{\alpha,\beta,\rho\}\subseteq\text{range}(i_{z(n+1)\cap\nu,\nu})$,

(c) $z(n) \subseteq \text{range}(i_{z(n+1)\cap\nu,\nu})$,
(d) $\text{range}(i_{z(n)\cap\nu,\nu}) \cap \rho \subseteq z(n+1)$.
Let $z' = \bigcup\{z(n) : n < \omega\}$, and it is easy to show that $z' \in C$, so $C$ is unbounded.

**4.13. Corollary.** *Let $\alpha$ be uncountable and regular and let $\nu \geqslant \alpha$ such that $M_\nu$ exists. Suppose $|\gamma| \leqslant |\nu|$ and define $f$ with domain $\mathcal{P}_\alpha\nu$ by*

$$f(x) = \begin{cases} \text{the unique } \delta \text{ such that } i_{x\nu}\delta = \gamma, \\ \text{if } M_x \text{ and such a } \delta \text{ both exist,} \\ 0, \quad \text{otherwise.} \end{cases}$$

*Then if* $\text{range}(f) \subseteq \alpha$, *$f$ is canonical and $\|f\| = \gamma$.*

**Proof.** $\gamma < \alpha$ is easy. If $\alpha \leqslant \gamma \leqslant \nu$, apply Lemma 4.12 with $\beta = \nu$ and $\rho = \gamma$ and use the fact that $f'(x) = \overline{x \cap \gamma}$ is canonical of rank $\gamma$. If $\gamma > \nu$, applying Lemma 4.12 with $\beta = \rho = \gamma$ gives that $f(x \cap \nu) = \bar{x}$ on a club subset of $\mathcal{P}_\alpha\gamma$. Since the function $g(x) = \bar{x}$ is canonical of rank $\gamma$ in $\mathcal{P}_\alpha\gamma$ and $|\gamma| = |\nu|$, the rest follows easily from the isomorphism $\iota_{\nu\gamma} : \mathcal{B}(\alpha, \nu) \to \mathcal{B}(\alpha, \gamma)$.

**4.14. Lemma.** *Suppose $M_\alpha$ exists, $\alpha$ is regular, and*
(a) *For all $\delta < \alpha$, there is an $\eta < \alpha$ such that $\eta \leqslant \gamma < \alpha$ implies $\kappa_\gamma > \delta$.*
(b) *For all $\gamma < \alpha$, $|\text{range}(i_{\gamma\alpha}) \cap \alpha| < \alpha$.*
*Then $A = \{\kappa_\gamma : \gamma < \alpha$ and $i_{\gamma\alpha}\kappa_\gamma = \alpha\}$ contains a club subset of $\alpha$.*

**Proof.** Applying Lemma 4.12 with $\alpha = \nu = \beta = \rho$ and then intersecting with $\alpha$ (which is club in $\mathcal{P}_\alpha\alpha$) gives that

$$C = \{\gamma < \alpha : \alpha \in \text{range}(i_{\gamma\alpha}) \text{ and } \gamma = \text{range}(i_{\gamma\alpha}) \cap \alpha\}$$

is club in $\mathcal{P}_\alpha\alpha$, and therefore in $\alpha$. Note that if $\gamma \in C$, then $i_{\gamma\alpha}\gamma = \alpha$, so for all $\gamma \in C$, there must be a $\delta$, $\gamma \leqslant \delta < \alpha$, such that $\kappa_\delta \leqslant \gamma$, for otherwise we would have $i_{\gamma\alpha}\gamma = \gamma$. Now let

$$T = \{\gamma < \alpha : \exists\delta\, (\kappa_\delta < \gamma \leqslant \delta < \alpha)\}.$$

We show $T$ is nonstationary, for suppose $T$ is stationary. Then $f : T \to \alpha$ defined by $f(\gamma) = \kappa_\delta$, where $\delta$ is least such that $\kappa_\delta < \gamma \leqslant \delta < \alpha$, is regressive, so cofinally many $\kappa_\delta$'s are the same, contradicting (a). Thus $C - T$ contains a club set, and clearly $C - T \subseteq A$.

**4.15. Definition.** Let $U$ be a coherent sequence as in Definition 4.6. Let $\theta$ be the least ordinal $> l^U$ such that $i_{\mathcal{U}}\theta = \theta$ for all $\mathcal{U} \in \text{range}(U)$, and let $\mu > \theta$ be any uncountable regular cardinal which is not weakly Mahlo ($\mu$ is fixed throughout the construction). The reason for the restrictions on $\mu$ is that we want to find an interval $I$ of ordinals (with least element $\mu$) and an iterated ultrapower $i : L[U] \to L[i(U)]$ such that $i(O^U)$ and $m^*$ agree on the interval $I$, with the smallest measurable cardinal of $L[i(U)]$ being $\mu$. Thus we need $(i(O^U))(\mu)=m^*(\mu)=\mu+1$. Define iterated ultrapowers $M_\alpha$ by induction on $\alpha$ with $M_0=L[U]$ and taking direct limits at limits. Suppose $M_\alpha$ has been defined.

*Case* I: $0<\alpha<\mu$. Then let $\kappa_\alpha$ be the least element of $\{\gamma: O_\alpha(\gamma)>\gamma\}$ and $\mathcal{U}_\alpha=(i_\alpha U)(\kappa_\alpha,\kappa_\alpha)$, where for convenience, $O_y=O^{i_yU}$ for any y such that $M_y$ exists.

*Case* II: $\alpha\geq\mu$. Then let $(\kappa,\lambda)$ be the $<_m$-least pair such that $(\kappa,\lambda)\in\mathrm{dom}(i_\alpha U)$ and $m^*(\kappa)=\lambda$. Let $\kappa_\alpha=\kappa$ and $\mathcal{U}_\alpha=(i_\alpha U)(\kappa,\lambda)$. Let $\lambda_\alpha=\lambda$. Case II continues throughout the ordinals until such time (if any) that no $(\kappa,\lambda)$ as specified exists. We let $\Omega=\{\alpha:\kappa_\alpha$ is defined$\}$. Thus $\Omega$ is either an ordinal or the class of all ordinals. From now on any reference to $\kappa_\alpha$, $i_\alpha$, $M_\alpha$, etc. will assume the item in question exists. Some basic properties of this iterated ultrapower are:

**4.16. Proposition.** (a) *If $M_y$ exists, then $|i_y\theta|\leq\max\{|y|,|\theta|\}$*

(b) *If $\alpha<\alpha'$, then $\lambda_\alpha<\lambda_{\alpha'}$ and $\kappa_\alpha\neq\kappa_{\alpha'}$.*

(c) *If $(\delta,m^*(\delta))\leq_m(\kappa_\alpha,\lambda_\alpha)$, then for all $\gamma>\alpha$, $O_\gamma(\delta)\leq m^*(\delta)$.*

(d) *If $\kappa_\tau\leq\gamma<\tau$, then $\kappa_\gamma>\kappa_\tau$.*

**Proof.** (a) By induction on $\bar{y}$, using the defining property of $\theta$ at successors. (b) and (c) are routine by induction on $\alpha$, using the coherence property (Definition 4.5(5)). Remark (1) after Definition 4.6 is useful in showing $\lambda_{\alpha+1}>\lambda_\alpha$.

(d) Suppose $\kappa_\tau\leq\gamma<\tau$ and $\kappa_\gamma\leq\kappa_\tau$. Then $m^*(\kappa_\gamma)=\lambda_\gamma\geq\gamma\geq\kappa_\tau\geq\kappa_\gamma$ and by Theorem 3.3, $\lambda_\gamma=m^*(\kappa_\gamma)\geq m^*(\kappa_\tau)=\lambda_\tau$, contradicting (b) since $\gamma<\tau$.

**4.17. Definition.** Let $\alpha\geq\mu$ be regular such that $M_\alpha$ exists and let $\beta\geq\alpha$. Let $\nu=\nu(\alpha,\beta)$ be the greatest $\nu$ such that $M_\nu$ exists and whenever $\alpha\leq\gamma<\nu$, we have $\lambda_\gamma<\beta\cap O_\gamma(\alpha)$. (Existence of $\nu(\alpha,\beta)$ is no problem, since $\alpha$ is vacuously a 'potential $\nu$' and the set of all 'potential $\nu$'s' is closed. Since the $\lambda_\gamma$'s are strictly increasing, it is easy to see that $\nu(\alpha,\beta)\leq\beta$).

$Q(\alpha,\beta)$ is defined to be the set of all $y\in\mathcal{P}_\alpha\nu$ satisfying all of the following properties.

(a) $M_y$ exists and $y\cap\alpha$ is an ordinal and $i_{y\nu}(y\cap\alpha)=\alpha$.

(b) $\kappa_{\bar{y}}=y\cap\alpha$.

(c) $i_{y\nu}(O_{y+1}(y\cap\alpha))=\beta$ (i.e. $i_{y\nu}(m^*(y\cap\alpha))=\beta$ if $\bar{y}\geq\mu$).

(d) For all $\gamma\in y$, $i_{y\cap\gamma,\gamma}\mathcal{U}_{\overline{y\cap\gamma}}=\mathcal{U}_\gamma$.

**4.18. Lemma.** *Let $\alpha,\beta,\nu$ be as in Definition 4.17, with $\beta\leq m^*(\alpha)$.*

(a) *If $\alpha\leq\gamma<\nu$, then $\kappa_\gamma>\alpha$.*

(b) *There is an ordinal $\delta<\alpha$ such that whenever $\sigma\subseteq y\in Q(\alpha,\beta)$, $\bar{y}<\gamma<\nu$ implies $\kappa_\gamma>\lambda_{\bar{y}}$.*

**Proof.** (a) Suppose not, i.e. $\kappa_\gamma\leq\alpha$. Then $\alpha\leq\lambda_\gamma<\beta\leq m^*(\alpha)$, so since $\lambda_\gamma=m^*(\kappa_\gamma)$, $\kappa_\gamma\neq\alpha$, so $\kappa_\gamma<\alpha$. But then $m^*(\kappa_\gamma)\geq m^*(\alpha)$, a contradiction.

(b) If $Q(\alpha,\beta)\neq\emptyset$, then since $M_{\bar{\gamma}}=M_\gamma$ (by 4.17(d)), $O_\gamma(\kappa_{\bar{\gamma}})=O_{\bar{\gamma}}(\kappa_{\bar{\gamma}})>\kappa_{\bar{\gamma}}$, so $O_\nu(\alpha)>\alpha$ and $\{\gamma<\alpha : O_\nu(\gamma)>\alpha\}$ is bounded below $\alpha$ by 4.6(a). Let $\sigma<\alpha$ be such that if $\gamma<\alpha$ and $O_\nu(\gamma)>\alpha$, then $i_{\sigma\nu}\gamma=\gamma$. We can do this by (a), since it is easy to find an $x\subseteq\nu$ such that $i_{x\nu}\gamma=\gamma$ for all such $\gamma$, in which case $x\cap\alpha$ also works. It is easy to see that $\sigma$ works.

The following lemma, which asserts a close relationship between the $Q(\alpha,\beta)$'s and the $Q^\alpha_\beta$'s defined in Section 3, is the main technical lemma of this section.

**4.19. Lemma.** *Let $\alpha$, $\beta$, $\nu$ be as in Definition* 4.17. *Then*

(a) $\iota_{\nu\beta}[Q(\alpha,\beta)]=[Q^\alpha_\beta]$.

(b) *If $m^*(\alpha)<\infty$ and $\nu=\nu(\alpha,m^*(\alpha))$, then $m^*(\alpha)\leqslant O_\nu(\alpha)$, and for all $\gamma>\nu$ such that $M_\gamma$ exists, $m^*(\alpha)=O_\gamma(\alpha)$.*

**Proof.** We prove by induction on $<_{\mathrm{m}}$, so suppose the lemma is true for all $\alpha'$, $\beta'$, $\nu'=\nu(\alpha',\beta')$ satisfying the hypothesis such that $(\alpha',\beta')<_{\mathrm{m}}(\alpha,\beta)$. If $\alpha=\mu$, it is easy to check that $\nu(\mu,\beta)=\nu$ for all $\beta\geqslant\alpha$, that $Q(\mu,\beta)=\emptyset$ for $\beta>\mu$, and that $Q(\mu,\mu)$ contains the club set $\{\gamma<\mu : \kappa_\gamma=\gamma\}$, and the rest is routine since $\mu$ was chosen to be regular but not weakly-Mahlo. Thus for the rest of the proof we assume $\alpha>\mu$. To prove (a) for $(\alpha,\beta)$, let $A=\{x\in\mathcal{P}_\alpha\beta : x\cap\nu\in Q(\alpha,\beta)\}$, and by definition of $\iota_{\nu\beta}$ we will be done if we can find a club $C\subseteq\mathcal{P}_\alpha\beta$ such that $A\cap C=Q^\alpha_\beta\cap C$.

We now define several club sets which we shall intersect to get $C$. For each $\gamma<\beta$ we define $C(\gamma)$ as follows: If $\gamma<\alpha$ or $\gamma>\nu$ we let $C(\gamma)=\mathcal{P}_\alpha\beta$. If $\alpha\leqslant\gamma\leqslant\nu$, let $\rho(\gamma)=\min\{\beta, i_\gamma\theta\}$ and let

$$C(\gamma)=\{x\in\mathcal{P}_\alpha\beta : M_{x\cap\gamma}\text{ exists, }\{\alpha,\beta,\rho(\gamma)\}\subseteq\operatorname{range}(i_{x\cap\gamma,\gamma}),\text{ and }\\ x\cap\rho(\gamma)=\operatorname{range}(i_{x\cap\gamma,\gamma})\cap\rho(\gamma)\}.$$

Using 4.16(a), it is easy to check that the hypotheses of Lemma 4.12 are satisfied, so $C(\gamma)$ is club. If $\nu=\beta$, define $C(\nu)$ the same way. Now let

$$x\in C_0 \quad\text{iff}\quad \text{for all } \gamma\in x,\ x\in C(\gamma)\cap C(\nu).$$

Let

$$C_1=\{x : x\cap\alpha \text{ is an ordinal }\geqslant\mu\text{ and } M_{x\cap\nu}\text{ exists}\},$$

$$C_2=\{x : x\cap\alpha\in B\},\text{ for some } B\subseteq\alpha \text{ club in } \alpha \text{ such that}\\ B\subseteq\{\kappa_\gamma : \gamma<\alpha \text{ and } i_{\gamma\alpha}\kappa_\gamma=\alpha\},$$

$$C_3=\{x : \text{if } \gamma\in x\cap\Omega,\text{ then } \{\kappa_\gamma,\lambda_\gamma\}\cap\beta\subseteq x\},$$

$$C_4=\{x : \text{if } \gamma\in x \text{ and } m^*(\gamma)<\beta,\text{ then } m^*(\gamma)\in x\}.$$

$C_2$ is club by Lemma 4.14 and for $C_1$, $C_3$, $C_4$ it is easy to verify the club property.

Let $C=C_0\cap C_1\cap C_2\cap C_3\cap C_4$. We now show that $A\cap C=Q^\alpha_\beta\cap C$.

($\subseteq$) Suppose $x \in A \cap C$. Let $y = x \cap \nu$, so by definition of $A$, $y \in Q(\alpha, \beta)$. Note that 4.17(d) implies that for all $\gamma \in y$, $M_{\overline{y \cap \gamma}} = M_{y \cap \gamma}$ and $i_{y \cap \gamma} = i_{\overline{y \cap \gamma}}$, and that $M_y = M_{\bar{y}}$, $i_y = i_{\bar{y}}$. Thus since $O_{\bar{y}+1}(y \cap \alpha) < i_{\bar{y}+1}\theta = i_{\bar{y}}\theta$, it is clear from 4.17(c) that $\beta < i_\nu \theta$, so $\rho(\nu) = \beta$. So since $x \in C(\nu)$, $i_{y\nu}\bar{x} = \beta$, i.e. $m^*(x \cap \alpha) = m^*(y \cap \alpha) = \bar{x}$, since we are considering, only y's such that $\mu \subseteq \gamma$. Thus $x$ satisfies (a) and (b) of Definition 3.2 for $Q^\alpha_\beta$. Now let $\gamma, \delta \in x$ with $\gamma \leqslant \delta$.

*Case* I: $Q^{\overline{x \cap \gamma}}_{\overline{x \cap \delta}}$ *is nonstationary*. Then $m^*(\overline{x \cap \gamma}) \leqslant \overline{x \cap \delta}$, so pick $\eta \in x$ such that $m^*(\overline{x \cap \gamma}) = \overline{x \cap \eta}$, and clearly $\gamma \leqslant \eta \leqslant \delta$. If $\eta < \alpha$, then clearly $Q^\gamma_\eta = Q^{\overline{x \cap \gamma}}_{\overline{x \cap \eta}}$ is nonstationary, so assume $\eta \geqslant \alpha$. Since $m^*(\overline{x \cap \gamma}) < \bar{x} = m^*(\overline{x \cap \alpha})$, clearly $\overline{x \cap \gamma} \neq \overline{x \cap \alpha} = \kappa_{\bar{y}}$, so from the definition, $\nu(\overline{x \cap \gamma}, \overline{x \cap \eta}) \leqslant \bar{y}$, so $\overline{x \cap \eta} = m^*(\overline{x \cap \gamma}) = O_{\bar{y}}(\overline{x \cap \gamma})$ by the induction hypothesis on (b), since $i_{\bar{y},\bar{y}+1}$ does not change $O_{\bar{y}}(\overline{x \cap \gamma})$. Thus $\overline{x \cap \eta} = O_y(\overline{x \cap \gamma})$, since $i_y = i_{\bar{y}}$. Applying $i_{y\nu}$ to both sides and using the fact that $i_{y\nu}\overline{x \cap \gamma} = \gamma$ and $i_{y\nu}\overline{x \cap \eta} = \eta$ (easy since $x \in C(\nu)$), we get $O_\nu(\eta) = \eta$. Since $\nu(\gamma, \eta) \leqslant \nu$, $m^*(\gamma) \leqslant O_\nu(\gamma) = \eta$, by the induction hypothesis on (b). Thus $Q^\gamma_\eta$ is nonstationary and therefore so is $Q^\gamma_\delta$.

*Case* II: $Q^\gamma_\delta$ *is nonstationary*. Let $\eta = m^*(\gamma) \leqslant \delta$. Again we can assume $\eta \geqslant \alpha$. It is easy to check that $\nu(\gamma, \eta) \leqslant \nu$, so by the induction hypothesis on (b), $m^*(\gamma) \leqslant O_\nu(\gamma)$, and it is easy to see that in fact equality holds, for if $m^*(\gamma) < O_\nu(\gamma)$, then either $\kappa_\nu = \gamma$ or $\kappa_\nu = \gamma'$ for some other $\gamma'$ with the same properties, violating the definition of $\nu(\alpha, \beta)$ as being the *greatest* $\nu$ such that ... etc. Since $O_\nu(\gamma) = \eta$, we argue as in Case I to get $O_{\bar{y}}(\overline{x \cap \gamma}) = \overline{x \cap \eta}$, so $m^*(\overline{x \cap \gamma}) \leqslant \overline{x \cap \eta}$, by choice of $\kappa_{\bar{y}}$, i.e. $Q^{\overline{x \cap \gamma}}_{\overline{x \cap \eta}}$ (and therefore $Q^{\overline{x \cap \gamma}}_{\overline{x \cap \delta}}$) is nonstationary.

Then $x \in Q^\alpha_\beta$ by Definition 3.2(c).

($\supseteq$) Suppose $x \in Q^\alpha_\beta \cap C$. Let $y = x \cap \nu$. Then 4.17(a) is immediate since $x \in C$. Since $i_{y\nu}\bar{x} = \beta$, as $x \in C(\nu)$, and $m^*(y \cap \alpha) = m^*(\overline{x \cap \alpha}) = \bar{x}$, 4.17(c) is easy. Note that by the definition of $C(\gamma)$, if $\gamma \in y \cup \{\nu\}$ and $\delta \leqslant \rho(\gamma)$, then $i_{x \cap \gamma, \gamma}(\overline{x \cap \delta}) = \delta$. We can prove that for all $\gamma \in y$, $i_{y \cap \gamma, \gamma}\mathcal{U}_{\overline{y \cap \gamma}} = \mathcal{U}_\gamma$. We do so by induction on initial segments of y, so let $\gamma \in y$ and assume that if $\eta \in y \cap \gamma$, $i_{y \cap \eta, \eta}\mathcal{U}_{\overline{y \cap \eta}} = \mathcal{U}_\eta$. Then by 4.17(d), $M_{y \cap \gamma} = M_{\overline{y \cap \gamma}}$. Thus we get that for all $\delta \in y$, $i_{y \cap \gamma, \gamma}(O_{\overline{y \cap \gamma}}(y \cap \delta)) = O_\gamma(\delta)$. By Definition 3.2 we have for all $\delta, \tau \in x \cap \rho(\gamma)$, $Q^\delta_\tau$ is stationary iff $Q^{\overline{x \cap \delta}}_{\overline{x \cap \tau}}$ is stationary, i.e. $m^*(\delta) \leqslant \tau$ iff $m^*(\overline{x \cap \delta}) \leqslant \overline{x \cap \tau}$. It is easy to see that $\lambda_\gamma < \rho(\gamma)$ (by definition of $\theta$, $O_\gamma(\delta) < i_\gamma \theta$ for all $\delta$) and that $\lambda_{\overline{x \cap \gamma}}$ is less than the preimage in $M_{x \cap \gamma}$ $(= M_{\overline{x \cap \gamma}})$ of $\rho(\gamma)$), so by the choices of $\kappa_\gamma$ and $\kappa_{\overline{x \cap \gamma}}$ we must have $i_{y \cap \gamma, \gamma}\mathcal{U}_{\overline{y \cap \gamma}} = \mathcal{U}_\gamma$. Not that 4.17(d) has been established, we see that $i_y = i_{\bar{y}}$ and $M_y = M_{\bar{y}}$. Since $x \in C_2$ we know that $y \cap \alpha = \kappa_\tau$ for some $\tau$, and it is easy to see that we must have $\tau = \bar{y}$, since if we had $\kappa_\tau \neq y \cap \alpha$, we could argue as we just did for proving (d) of 4.17. (using $i_{y\nu}$ instead of $i_{y \cap \gamma, \gamma}$) that $\kappa_\nu$ would be chosen so that $\alpha < \kappa_\nu < O_\nu(\alpha)$ and $\lambda_\nu < \beta$, violating the definition of $\nu(\alpha, \beta)$. Thus $\kappa_{\bar{y}} = y \cap \alpha$ and $x \in A$, so we have proven (a) of the Lemma.

To prove (b), we only need to take a closer look at the case $\beta = O_\nu(\alpha)$, where $\nu = \nu(\alpha, m^*(\alpha))$. If $m^*(\alpha) \leqslant \beta$ we are done, so assume $m^*(\alpha) > O_\nu(\alpha)$. Using the definition of $\nu(\alpha, \beta)$, it is easy to check that $\nu(\alpha, \beta) = \nu$. We will show that $Q(\alpha, \beta) = \emptyset$, contradicting $m^*(\alpha) > O_\nu(\alpha)$, since $Q(\alpha, \beta)$ should be stationary by (a) of the lemma.

Suppose $y \in Q(\alpha, \beta)$. Then $i_y = i_{\bar{y}}$ by 4.17(d) and 4.10(d). Thus

$$i_{y\nu}(O_{\bar{y}+1}(y \cap \alpha)) = \beta = O_\nu(\alpha) = i_{y\nu}(O_y(y \cap \alpha)) = i_{y\nu}(O_{\bar{y}}(y \cap \alpha)),$$

so $O_{\bar{y}}(y \cap \alpha) = O_{\bar{y}+1}(y \cap \alpha)$, contradicting $\kappa_{\bar{y}} = y \cap \alpha$. Thus $m^*(\alpha) \leq O_\nu(\alpha)$. By choice of $(\kappa_\nu, \lambda_\nu)$, we must have $\lambda_\nu \geq m^*(\alpha)$, for it is easy to see that $\lambda_\nu < m^*(\alpha)$ would violate the definition of $\nu(\alpha, m^*(\alpha))$.

**4.20. Corollary.** *Suppose $\Omega$ is an ordinal. Then*

$$\operatorname{dom}(U_{\Omega+1}) = \operatorname{dom} \mathcal{F}_\mu \restriction \Omega + 1.$$

**Proof.** We need to show that for all $\alpha \in [\mu, \Omega+1)$, $m^*(\alpha) = O_{\Omega+1}(\alpha)$, and that $O_{\Omega+1}(\alpha) = \alpha$ whenever $\alpha \geq \Omega + 1$. Note that for all $\alpha \geq \mu$, $\alpha \leq \Omega$, $\nu(\alpha, \beta) \leq \Omega$ trivially, so it is easy to show that $Q(\alpha, i_\Omega\theta) = \emptyset$, so that $m^*(\alpha) < \infty$ by Lemma 4.19(a), so $m^*(\alpha) = O_{\Omega+1}(\alpha)$ by 4.18(b). (The case where $\alpha$ is singular is trivial.) Suppose there were an $\alpha \geq \Omega + 1$ such that $O_{\Omega+1}(\alpha) > \alpha$. Then by definition of $\theta$ we would have $\alpha < i_\Omega\theta$. But $|i_\Omega\theta| = |\Omega|$, so $\alpha$ is not regular and $m^*(\alpha) = \alpha < O_{\Omega+1}(\alpha)$. But then $(\alpha, m^*(\alpha))$ would be a possible choice for $(\kappa_\Omega, \lambda_\Omega)$, contradicting the definition of $\Omega$.

To prove Theorem 4.8 we only need to show that $L[U_{\Omega+1}] = L[\mathcal{F}_\mu \restriction \Omega + 1]$. The above corollary gives part of that result, showing that the domains at least match. For the rest we need to show that for all $(\alpha, \beta) \in \operatorname{dom}(U_{\Omega+1})$, $U_\Omega(\alpha, \beta)$ '$\subseteq$' $\mathcal{F}_\mu(\alpha, \beta)$. The quotes are there because the containment is only true if each $X \in U_\Omega(\alpha, \beta)$ is identified with the appropriate $X' \subseteq^{(\operatorname{supp} X)}\alpha$.

**4.21. Definition.** Suppose $\alpha \geq \mu$ such that $M_\alpha$ exists and $\alpha \leq \beta < m^*(\alpha)$. Let $\nu = \nu(\alpha, \beta)$ and $\delta = (i_\nu\delta)(\alpha, \beta)$. (See Definition 4.5.) Let $\mathcal{U} = i_\nu U(\alpha, \beta)$ and let $[\cdot]_{\mathcal{U}}$ indicate the ultrapower taken inside $M_\nu$. (Note that $\mathcal{U} \subseteq \mathcal{P}_f({}^\delta\alpha)$.) Define the function $b_{\alpha\beta} \in M_\nu$ with domain $\delta$ as follows: For each $\eta < \delta$, let $b_{\alpha\beta}(\eta) = [(a_\eta : a \in {}^\delta\alpha)]_{\mathcal{U}}$. For all such $(\alpha, \beta)$ we define a filter $\mathcal{G}(\alpha, \beta) \subseteq \mathcal{P}_f({}^\delta\alpha)$. We first define a $G \subseteq \mathcal{P}_f({}^\delta\mathcal{P}_\alpha\nu)$: We let $G[1]$ be the filter generated by $Q(\alpha, \beta)$. If $G[\eta]$ has been defined and $\eta < \delta$ we let $b_{\alpha\beta\eta} : \mathcal{P}_\alpha\nu \to \alpha$ be the canonical function of rank $b_{\alpha\beta}(\eta)$ defined by $i_{x\nu}b_{\alpha\beta\eta}(x) = b_{\alpha\beta}(\nu)$ whenever this makes sense (Corollary 4.13). That range$(b_{\alpha\beta\eta}) \subseteq \alpha$ is easy to show since $b_{\alpha\beta}(\eta) < i_\nu\theta$. Then let

$$G[\eta+1] = \{X \in \mathcal{P}_f({}^{\eta+1}\mathcal{P}_\alpha\nu) : \{a \restriction \eta : a \in X \text{ and } b_{\alpha\beta\eta}(a_0) = \overline{a_\eta \cap \alpha}\} \in G[\eta]\}.$$

At limits we take the 'union'. As in Definition 4.7, we then let $\mathcal{G}(\alpha, \beta) = \{X^* : X \in \mathcal{G}\}$. Thus, the definition here is essentially the same as in Definition 4.7 for $\mathcal{F}_\mu$ except that we use $b_{\alpha\beta}(\eta)$ instead of $\mathrm{b}_\eta$ and $Q(\alpha, \beta)$ instead of $Q^\alpha_\beta$.

The proof of the Main Theorem will proceed in two steps. We first prove that for all appropriate $(\alpha, \beta)$, $(i_\Omega U)(\alpha, \beta) \subseteq \mathcal{G}(\alpha, \beta)$ and then that $\mathcal{G}(\alpha, \beta) \subseteq \mathcal{F}_\mu(\alpha, \beta)$.

We first note a couple of easy properties of the functions $b_{\alpha\beta}$. Recall that if $\mathcal{U}$ is a countably complete ultrafilter over some index set $I$ and id is the identity function on $I$, then

(a) For all $X \subseteq I$, $X \in \mathcal{U}$ iff $[\mathrm{id}]_{\mathcal{U}} \in i_{\mathcal{U}} X$

(b) For all $f$ with domain $I$, $[f]_{\mathcal{U}} = (i_{\mathcal{U}})([\mathrm{id}]_{\mathcal{U}})$. However, when we consider $\mathcal{P}_f({}^{\delta}\alpha)$, the identity function does not have finite support unless $\delta$ is finite. The following proposition states the corresponding facts for $\mathcal{P}_f({}^{\delta}\alpha)$. Rather than giving a general statement we will state in the form in which it will be used.

**4.22. Proposition.** *Let $\alpha, \beta, \nu, \delta, \mathcal{U}, b_{\alpha\beta}$ be as in Definition 4.21. Let $i: M_\nu \to \mathrm{Ult}(M_\nu, \mathcal{U})$. Then*

(a) *For all $X \in \mathcal{P}_f({}^{\delta}\alpha) \cap M_\nu$, $X \in \mathcal{U}$ iff there is a $b \in iX$ with $\mathrm{supp}(b) \subseteq \mathrm{supp}(iX)$ such that for all $i(\eta) \in \mathrm{supp}(iX)$, $b_{i(\eta)} = b_{\alpha\beta}(\eta)$.*

(b) *If $f \in M_\nu$ with finite support and $b \in \mathrm{range}(i)$ such that $\mathrm{supp}(b) \subseteq \mathrm{supp}(if)$ and for $i(\eta) \in \mathrm{supp}(if)$, $b_{i(\eta)} = b_{\alpha\beta}(\eta)$, then $[f]_{\mathcal{U}} = (if)(b)$.*

**4.23. Lemma.** *Suppose $\mu \leq \alpha \leq \beta < m^*(\alpha)$, $M_\alpha$ exists, $\nu = \nu(\alpha, \beta)$. Then $(i_\nu U)(\alpha, \beta) \subseteq \mathcal{G}(\alpha, \beta)$.*

**Proof.** Let $\mathcal{U} = (i_\nu U)(\alpha, \beta)$, $G^* = \mathcal{G}(\alpha, \beta)$, and let $G$ be the filter on $\mathcal{P}_f({}^{\delta}\mathcal{P}_\alpha \nu)$ of Definition 4.21 (so that $G^* = \{X^* : X \in G\}$). Suppose $X \in \mathcal{U}$. Let $s = \mathrm{supp}\, X \cup \{0\}$, say $s = \{\eta_k : k < n\}$ with $\eta_k = 0$ and $\eta_k < \eta_{k+1}$. Let $C \subseteq \mathcal{P}_\alpha \nu$ be club such that for all $x \in X$, $\{b_{\alpha\beta}, s, X\} \subseteq \mathrm{range}(i_{x\nu})$, and $\sigma \subseteq x$ where $\sigma$ is as in Lemma 4.18(b). Define $Y \subseteq {}^{\delta}\mathcal{P}_\alpha \nu$ with support in $s$ by $a \in Y$ iff

(i) $a_0 \in C \cap Q(\alpha, \beta)$, and

(ii) for all $k < n$, $b_{\alpha\beta\eta_k}(a_0) = \overline{a_{\eta_k} \cap \alpha}$.

It is easy to check from the definitions that $Y \in G$, so $Y^* = \{a^* : a^* \in Y\} \in G^*$. Thus, since $G^*$ is a filter, we are done if we can show that $Y^* \subseteq X$, i.e. we need $a \in Y$ implies $a^* \in X$. Suppose $a \in Y$. Let $y = a_0$, so $y \in Q(\alpha, \beta) \cap C$. Let $X', s'$ be such that $i_{y\nu} X' = X$, $i_{y\nu} s' = s$ and suppose $s' = \{\eta'_k : k < n\}$. Note that 4.17(b), (c) imply that $\kappa_{\bar{y}} = y \cap \alpha$ and that $i_{y\nu} \mathcal{U}_{\bar{y}} = \mathcal{U}$ (recall that $M_{\bar{y}} = M_y$ by 4.17(d)). Thus $X' \in \mathcal{U}_{\bar{y}}$ because $X \in \mathcal{U}$. Using 4.22(a) traced back to $M_y$ via $i_{y\nu}$, we get that if $b \in M_y$ is defined by $b(i_{\bar{y},\bar{y}+1} \eta'_k) = b_{\alpha\beta\eta_k}(a_0)$ for $k > 0$ and $b(0) = \overline{y \cap \alpha}$, with $b(\eta)$ arbitrary for other $\eta$, then $b \in i_{\bar{y},\bar{y}+1} X'$, since $\mathcal{U}_{\bar{y}}$ was used to form $i_{\bar{y},\bar{y}+1}$. Thus $i_{\bar{y}+1}$, $b \in i_{\bar{y}\nu} X'$. Now $i_{\bar{y}\nu} \neq i_{y\nu}$ in general, but $i_{\bar{y}\nu} X'$ is essentially the 'same' as X except that its support is $i_{\bar{y}\nu} s'$ instead of $s$. But $i_{\bar{y}+1,\nu}$ has the same relation to $a^*$, i.e., for all $\eta \in s'$,

$$(i_{\bar{y}+1,\nu} b)(i_{\bar{y}\nu} \eta' k) = i_{\bar{y}+1,\nu} b_{\alpha\beta\nu_k}(a_0) \overset{*}{=} b_{\alpha\beta\nu_k}(a_0) = a^*_{\eta_k}.$$

The equality marked '*' holds by Lemma 4.18(b), since $b_{\alpha\beta\nu_k}(a_0) < \lambda_{\bar{y}}$ (to see this apply $i_{y\nu}$ to both sides, which gives $b_{\alpha\beta}(\nu) < \beta$, an inequality that is immediate from Definition 4.5(1) and the definition of $b_{\alpha\beta}$). Thus since $i_{\bar{y}+1,\nu} b \in i_{\bar{y}\nu} X'$ we must have $a^* \in X$.

**4.24. Corollary.** *Suppose $\mu \leq \alpha \leq \beta < m^*(\alpha)$, $M_\alpha$ exists, $\nu = \nu(\alpha, \beta)$. Then for all $\gamma$ such that $\nu \leq \gamma \leq \Omega$,*

$$(i_\gamma U)(\alpha, \beta) \subseteq \mathcal{G}(\alpha, \beta).$$

**Proof.** $(\kappa_\gamma, \lambda_\gamma) >_{\mathrm{m}} (\alpha, \beta)$ for $\gamma \geqslant \eta$, so the coherence condition guarantees that $(i_\nu U)(\alpha, \beta)$ remains unchanged.

**4.25. Lemma.** *Suppose $\mu \leqslant \alpha \leqslant \beta < m^*(\alpha)$ and $M_\alpha$ exists.*
*Then $\mathcal{G}(\alpha, \beta) \subseteq \mathcal{F}_\mu(\alpha, \beta)$.*

**Proof.** By induction on $<_{\mathrm{m}}$. Suppose $\mathcal{G}(\alpha', \beta') \subseteq \mathcal{F}_\mu(\alpha', \beta')$ for all $(\alpha', \beta') <_{\mathrm{m}} (\alpha, \beta)$ such that $(\alpha', \beta') \in \operatorname{dom} \mathcal{G}$. Then by Corollary 4.24, $(i_\nu U)(\alpha', \beta') \subseteq \mathcal{F}(\alpha', \beta')$ for all such $(\alpha', \beta')$, where $\nu = \nu(\alpha, \beta)$. Thus $L[(i_\nu U) \restriction (\alpha, \beta)] = L[\mathcal{F}_\mu \restriction (\alpha, \beta)]$ (call this model $M$ for short). Note that $\mathcal{F}_\mu(\alpha, \beta)$ and $\mathcal{G}(\alpha, \beta)$ were defined (Definitions 4.7 and 4.21 respectively) in essentially the same way with only two differences. The first difference is that $\mathcal{F}_\mu(\alpha, \beta)$ was defined using $Q^\alpha_\beta \subseteq \mathcal{P}_\alpha \beta$, whereas $\mathcal{G}(\alpha, \beta)$ was defined using $Q(\alpha, \beta) \subseteq \mathcal{P}_\alpha \nu$. But since $\iota_{\nu\beta}[Q(\alpha, \beta)] = [Q^\alpha_\beta]$, this poses no problem, since everything in the definitions was intersected with $\alpha$ at the last moment. The other difference is that we used $\mathfrak{b}_\eta$ to define $F[\eta + 1]$ from $F[\eta]$ (with $F$ as in Definition 4.7) and we used $b_{\alpha\beta}(\eta)$ to define $G[\eta + 1]$ from $G[\eta]$ (with $G$ as in Definition 4.21). For the remainder of this proof abbreviate $b = b_{\alpha\beta}$ and $b_{\alpha\beta\eta} = b_\eta$.

We now show that $b(\eta) = {}_\eta$ for all $\eta$ by induction on $\eta$. Suppose $b \restriction n = \mathfrak{b} \restriction \eta$. We suppose $b(\eta) \neq \mathfrak{b}_\eta$ and get a contradiction.

*Case* I: $b(\eta) < \mathfrak{b}_\eta$. Then by definition of $\mathfrak{b}_\eta$ there is a $g \in M$ with finite support in $\eta$ and an $f : \mathcal{P}_\alpha \beta \to \alpha$ canonical of rank $b(\eta)$ such that $\{a : g(a^*) = f(a_0)\} \in F[\eta]$. Thus since $b \restriction n = \mathfrak{b} \restriction \eta$, $G[\eta]$ and $F[\eta]$ are 'essentially the same' (one is a filter on ${}^\eta\mathcal{P}_\alpha \nu$, the other on ${}^\eta\mathcal{P}_\alpha \beta$), so if we take any function with domain $\mathcal{P}_\alpha \nu$ that is canonical of rank $b(\eta)$, say $b_\eta = b_{\alpha\beta\eta}$, we have that $\{a : g(a^*) = b_\eta(a_0)\} \in G[\eta]$. Let

$$X = \{a : g(a^*) = a^*{}_\eta (= \overline{a_\eta \cap \alpha})\} \subseteq {}^{\eta+1}\mathcal{P}_\alpha \nu.$$

Then

$$\{a \restriction \eta : a \in X \text{ and } b_\eta(a_0) = \overline{a_\eta \cap \alpha}\} = \{a \in {}^n\mathcal{P}_\alpha \nu : g(a^*) = b_\eta(a_0)\} \in G[\eta],$$

so $X \in G[\nu + 1]$ by definition, so $X^* \in G$. But $X^* \in M$, so $X^* \in (i_\nu U)(\alpha, \beta)$. Since $X^* = \{a : g(a) = a_\eta\}$, $g \in M = L(i_\nu U) \restriction (\alpha, \beta)]$, and $g$ has support in $\eta$, which violates Definition 4.5(3).

*Case* II: $\mathfrak{b}_\eta < b(\eta)$. Let $\mathcal{U} = (i_\nu U)(\alpha, \beta)$ and pick $h \in M_\nu$ such that $[h]_{\mathcal{U}} = \mathfrak{b}_\eta$. Then $\{a : h(a) < a_\eta\} \in \mathcal{U}$, since $b(\eta) = [(a_\eta)]_{\mathcal{U}}$, so by Definition 4.5(2) there is a $g \in L[(i_\nu U) \restriction (\alpha, \beta)] = M$ with support in $\eta$ such that $\{a : g(a) = h(a)\} \in \mathcal{U}$, i.e. $[g]_{\mathcal{U}} = \mathfrak{b}_\eta$. Many of the details are now much like the proof of Lemma 4.23. We let

$$s = \operatorname{supp}(g) \cup \{0\} = \{\eta_k : k < \eta\},$$

let $C$ be a club set such that $\{b, s, g\} \subseteq \operatorname{range}(i_{x\nu})$ and $\sigma \subseteq x$ for all $x \in C$, where $\sigma$ is as in Lemma 4.18(b). By Corollary 4.13 we may assume that for all $x \in C$, $i_{x\nu} f(x) = \mathfrak{b}_\eta$ for some $f$ canonical of rank $\mathfrak{b}_\eta$. $Y \in \mathcal{P}_{\mathrm{f}}({}^n\mathcal{P}_\alpha \nu)$ is then defined by $a \in Y$ iff $a_0 \in Q(\alpha, \beta) \cap C$ and $a_{\eta_k} = b_{\eta_k}(a_0)$ for all $k < n$, and clearly $Y \in G[\nu]$. Let $a \in Y$, let $y = a_0$ and let $g'$, $s' \in M_y$ such that $i_{y\nu}s' = s$ and $i_{y\nu}g' = g$. Then since $y \in Q(\alpha, \beta)$, $i_{y\nu}\mathcal{U}_{\bar{y}} = \mathcal{U}$, so $\underline{[g']_{\mathcal{U}_y}} = f(y)$, so if $d \in M_y$ is defined by $d(i_{\bar{y},\bar{y}+1}\eta'_k) = b_{\eta_k}(a_0)$ for $k > 0$ and $d(0) = y \cap \alpha$, then $(i_{\bar{y},\bar{y}+1}g')(d) = f(y)$ by tracing back Proposition 4.22 from $M_\nu$ to $M_y$ using $i_{y\nu}$ (and the fact that $M_y = M_{\bar{y}}$). We now apply $i_{y+1,\nu}$ to both sides and use exactly the same argument regarding the supports of $i_{\bar{y}\nu}g'$ and $i_{y\nu}g' = g$ as was used for $i_{\bar{y}\nu}X'$ and X in Lemma 4.23. This gives $g(a^*) = f(y) = f(a_0)$. Since this was true for all $a \in Y$, we see that $\{a : g(a^*) = f(a_0)\} \in G[\nu]$. But, as remarked in Case I, $G[\nu]$ and $F[\nu]$ are 'essentially the same', so if $f' : \mathcal{P}_\alpha\beta \to \alpha$ is canonical of rank $\mathfrak{b}_\eta$, e.g. $f'(x) = f(x \cap \nu)$, then $\{a : g(a^*) = f'(a_0)\} \in F[\nu]$. But this violates the definition of $\mathfrak{b}_\eta$ in Definition 4.7.

Thus we have shown that $\mathfrak{b}_\eta = b(\eta)$. The fact that $\overline{a_0 \cap \mathfrak{b}_\eta}$ was used in Definition 4.7 to get $F[\eta+1]$ from $F[\eta]$, whereas $b_{\alpha\beta\eta}(a_0)$ was used in the corresponding place to get $G[\eta+1]$ from $G[\eta]$, is not important, for both $a_0 \mapsto \overline{a_0 \cap \mathfrak{b}_\eta}$ and $a_0 \mapsto b_{\alpha\beta\eta}(a_0)$ are canonical of rank $\mathfrak{b}_\eta = b(\eta)$.

There is one more thing to take care of: $b(\eta)$ was defined for all $\eta < (i_\nu\delta)(\alpha, \beta)$, whereas we stopped defining filters $F[\eta]$ when we reached a point such that $\mathfrak{b}_\delta \geq \beta$ for some $\delta \geq 1$. Thus we must show that $\delta = (i_\nu\delta)(\alpha, \beta)$, where $\delta \geq 1$ was the least such that $\mathfrak{b}_\delta \geq \beta$. If $\delta < (i_\nu\delta)(\alpha, \beta)$, then we would have $b(\delta) = \mathfrak{b}_\delta \geq \beta$, which is easily seen to be impossible by the definition of $b$ $(= b_{\alpha\beta})$ and Definition 4.5(1), whereas if $\delta > (i_\nu\delta)(\alpha, \beta) = \delta'$, then we have $\mathfrak{b}_{\delta'} < \beta$. Letting $[h]_{\mathcal{U}} = \mathfrak{b}_{\delta'}$ for some $h$ we see from Definition 4.5(5) (since $\beta = [a \mapsto O_\nu(a_0)]_{\mathcal{U}}$) that $[g]_{\mathcal{U}} = \mathfrak{b}_{\delta'}$ for some $g \in L[(i_\nu U)(\alpha, \beta)] = M$, and the rest of the argument would proceed as in Case II to get exactly the same contradiction. Thus $\delta = \delta'$ and we are done.

We now give the promised precise statement of the Main Theorem 4.8 and its proof:

**Theorem.** *If $\Omega$ is an ordinal, then there is an elementary embedding $i : L[U] \to L[\mathcal{F}_\mu \restriction \Omega + 1]$.*

**Proof.** It is immediate from Lemmas 4.24 and 4.25 that $L[i_\Omega U] = L[\mathcal{F}_\mu \restriction \Omega + 1]$.

We briefly remark on what the statement "enough high degree Mahlo cardinals" meant when we originally stated Theorem 4.8. Suppose there are no weakly Mahlo cardinals above $\mu$. Then since $O_\Omega$ and $m^*$ are supposed to match in the interval $[\mu, \Omega + 1)$, the process must go on forever. On the other hand, it is easy to see that if $U$ satisfies the restriction ($\alpha < \gamma \leq O^U(\alpha)$ implies $O^U(\gamma) = \gamma$) of Definition 4.5, then $[\mu, \Omega + 1)$ cannot contain a greatly Mahlo cardinal, for otherwise $O_\Omega$ and $m^*$ could not match. Thus, a proper class of greatly Mahlo cardinals suffices to take care of *all* sequences $U$ satisfying that restriction. Similarly, a proper class of regular cardinals $\kappa$ such that $\{\alpha < \kappa : m^*(\alpha) \geq \kappa\}$ is unbounded in $\kappa$ suffices to take care of all $U$ satisfying Definition 4.6. We also note that in many set-theoretical universes there will be many $U$'s such that

$\theta_U < \omega_1$, in which we can let $\mu = \omega_1$, and $O_\Omega$ and $m^*$ will then match on an initial segment of the ordinals.

Finally, we note that the filter sequence $\mathcal{F}_\mu$ was defined using certain $\mathcal{P}_\kappa\lambda$'s, which is suggestive of supercompact cardinals (none of the models L[U] in this paper have even a $\kappa^+$-supercompact cardinal). It remains to be seen whether this is a coincidence or whether it is related to ghost properties of supercompactness that the cardinals in $L[U]$ might have.

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